2023 Wits Mathematics Competition
Qualifying Round
Senior Secondary

Instructions

This exam consists of 20 multiple choice questions. There is one correct answer to each question. There is no penalty for incorrect answers. The mark allocation is as follows:

Questions 1-5 are each worth 3 points,
Questions 6-10 are each worth 4 points,
Questions 11-15 are each worth 5 points,
Questions 16-20 are each worth 6 points.
The total number of points available is 90.

The time limit on this exam is 75 minutes, calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed. If you are using the computer-friendly answer sheet you should fill it in in BLACK pen (other colours do not scan well). Time may be given for filling in name, school and other personal details.

"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?". — David Hilbert

A. 3 point questions

1. Solve $2x - 4 = 0$.

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

   Solution: The steps are outlined below;

$$\begin{aligned} 2x - 4 &= 0 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

   So the answer is **C**

2. Which square is closest to 2023?

   (A) 23^2 (B) 33^2 (C) 41^2 (D) 45^2 (E) 1203^2

   Solution: Let us round each number to the nearest ten and then square it. So we get the following;

$$\begin{aligned} 23 &\approx 20 & 20^2 &= 400 \\ 33 &\approx 30 & 30^2 &= 900 \\ 41 &\approx 40 & 40^2 &= 1600 \\ 45 &\approx 50 & 50^2 &= 2500 \\ 1203 &\approx 1200 & 1200^2 &= 1440000 \end{aligned}$$

   So the closest square will be 45^2. So the answer is **D**

3. Twelve friends met up at a bakery. On average they ate 1.5 muffins. None of them ate more than two muffins and two ate nothing. How many of them ate two muffins?

   (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

   Solution: If 2 ate nothing then only 10 people ate any muffins. Since no one ate more than 2 muffins the most amount of muffins they could've eaten together was 20. Then our average muffins eaten would be $\frac{20}{12} = 1\frac{2}{3} = 1.666... > 1.5$. So not all the 10 people ate 2 muffins. Since 1.666... is not that much bigger than 1.5 lets try just making 1 person eat 1 muffin and the other 9 eat 2 muffins. Now our average is $\frac{19}{12} = 1.5833... > 1.5$. Again our average is very close but still larger than 1.5 so lets again try making one less person eat 2 muffins. So now 8 people eat 2 muffins and 2 people eat 1 muffin. Then our average is $\frac{18}{12} = 1.5$. So the answer is **E**

4. 64 identical circles are each inscribed inside separate squares each with a perimeter of 1. The total area covered by all these circles would be?

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{3\pi}{2}$ (D) 2π (E) $\frac{5\pi}{2}$

Solution: If the perimeter of a square is 1 then the length of each side is $\frac{1}{4}$ and so the radius of the circle is $\frac{1}{8}$. The area of a circle is πr^2 and so the area of one of these circles is $\pi(\frac{1}{8})^2 = \frac{\pi}{64}$. If we 64 of these circles then the area covered by all these circles is $64 \times \frac{\pi}{64} = \pi$.
So the answer is **B**

5. What is the size of the largest set of consecutive four-digit numbers with at least one odd digit each?

(A) 800 (B) 1111 (C) 1115 (D) 2023 (E) 5000

Solution: Clearly all the numbers between **1**000 and 2000 all contain an odd digit since 1 is an odd number and will always be in the thousands place. Similarly all the numbers between **3**000 and 4000, **5**000 and 6000 and so on, all contain an odd digit.
Lets focus on **3**000 and 4000. Using the same idea we can extend this interval backwards to 2**9**00 since the 9 in the hundreds place will ensure that all the numbers between 2**9**00 and 3000 will have an odd digit. But we can push the interval back again to 28**9**0 and again the 9 in the tens place ensures that all the number between 28**9**0 and 2900 will have an odd digit. Finally we can push the interval back one step to 288**9**. So the final interval is 2889 to 4000. Thus the number of 4 digit numbers in this interval is $4000 - 2889 = 1111$.
So the answer is **B**

B. 4 point questions

6. $4^2 = 4 \times 4 = 16$ ends in a 6.

   $4^3 = 4 \times 4 \times 4 = 64$ ends in a 4.

   What does 4^{2023} end in?

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 0

   Solution: Let's look at a few more powers of 4;

   $4^1 = 4$ ends in a 4

   $4^2 = 4 \times 4 = 16$ ends in a 6.

   $4^3 = 4 \times 4 \times 4 = 64$ ends in a 4.

   $4^4 = 4 \times 4 \times 4 \times 4 = 256$ ends in a 6

   From this we can see that there is a pattern. When the power is odd then the last digit is 4 and when the power is even the last digit is 6. Since 2023 is odd the last digit is 4. So the answer is **B**

7. A painter takes two days to paint a room (all four walls and the ceiling). If he works at the same pace, how many days will he take to paint a room that is twice as wide, twice as long, and twice as high?

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 16

   Solution: The original room has 5 surfaces that are painted which are rectangles. If we double the length and height of the original wall to make a new wall then the new wall will be 4 times the size or 4 rectangles. But we have 4 walls so in the new room we have 16 rectangles ($4 \times 4 = 16$). If we do the same with ceiling, doubling the length and width, the ceiling is 4 times bigger or 4 rectangles. There is only one ceiling. So the total rectangles the painter will need to paint in the new room is $16 + 4 = 20$ rectangles. Since $20 \div 5 = 4$ there are 4 times as many original surfaces to paint, so if it took 2 days to paint the original 5 surfaces it will take $4 \times 2 = 8$ days to paint the 20 of the original surfaces.
   So the answer is **D**

8. What is the smallest natural number which has exactly six positive factors, three of which are odd and three of which are even?

   (A) 1 (B) 8 (C) 12 (D) 18 (E) 24

   Solution: First we know that 1 and the number itself are always factors. Second we know that the product of 2 even numbers is even and the product of 2 odd numbers is odd. This tells us that the odd numbers must be multiplied by the even number to get

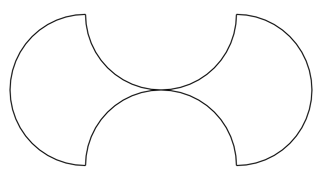
our number. So our number is even since 1 is odd.
Simply using a process of elimination at this point will be fastest.
1 is odd so it cannot be our number.
8 only has 4 factors so it cannot be our number.
12 has 6 factors but $2 \times 6 = 12$ and for our number we can't have an even number multiplied by even number. So 12 cannot be our number.
18 has 6 factors and they are $1, 2, 3, 6, 9$ and 18 so 18 fits all the criteria.
So the answer is **D**

9. The product of the integer ages of four childrens' ages is 450. No child's age is strictly greater than 18 or strictly younger than 1. (Here strictly means 1 and 18 are allowable values.) What is the minimum possible sum of their ages?

(A) 19 (B) 21 (C) 25 (D) 29 (E) 33

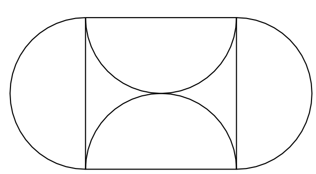
Solution: First we break 450 into its prime factors; $450 = 2 \times 3^2 \times 5^2$. So we have 5 prime factors that make 450 but only 4 children. We need to multiply two of the prime factors together to get one of the children's age. Since we want the minimum sum of the ages we want to multiply the two smallest prime factors together. So the ages of the children are $2 \times 3 = 6$ and $3, 5, 5$. That makes the sum; $3 + 5 + 5 + 6 = 19$.
So the answer is **A**

10. The following figure is a combination of four semi-circles, each with a radius of 3 cm. Calculate the figure's area.



(A) 18π (B) $18 + 9\pi$ (C) $36 - 9\pi$ (D) 36 (E) $36 + 4\pi$

Solution: Lets draw in a square on the diagram.



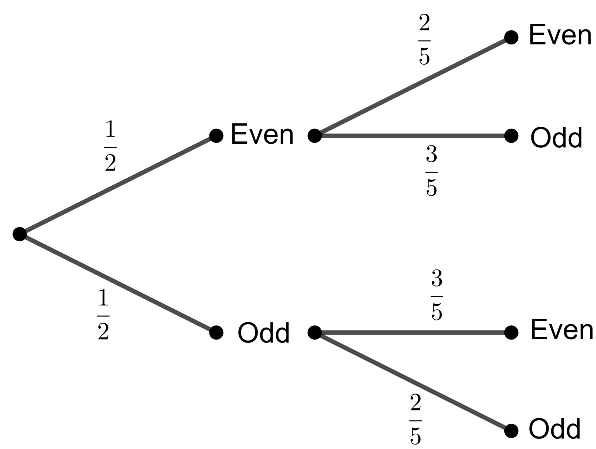
So since the radius of all the circles are 3 cm then the square has sides of 6 cm. The area of the square is $6^2 = 36$. The area of the semi-circle is $\frac{1}{2} \times \pi(3)^2 = \frac{9\pi}{2}$. So the area we want from inside the square is $36 - 2(\frac{9\pi}{2}) = 36 - 9\pi$. Now to calculate the area of the figure we add the area of the two remaining semi-circles, so we get $36 - 9\pi + 2(\frac{9\pi}{2}) = 36$.
So the answer is **D**

C. 5 point questions

11. Two distinct numbers are selected from the set of $\{1, 2, 3, 4, 5, 6\}$. Given that the product of these two numbers is even, what is the probability of their sum being even as well?

A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{2}{9}$ (E) $\frac{1}{2}$

Solution: Lets draw a tree diagram to capture the probabilities.



To get an even number from a product of 2 numbers at least one of the numbers must be even. So the branches Even-Even, Even-Odd and Odd-Even give us this. So the probability of an even number from the product is $(\frac{1}{2} \times \frac{2}{5}) + (\frac{1}{2} \times \frac{3}{5}) + (\frac{1}{2} \times \frac{3}{5}) = \frac{1}{5} + \frac{3}{10} + \frac{3}{10} = \frac{8}{10}$.
For the sum of 2 numbers to be even both numbers must be even. So the branch Even-Even is the only one that gives us this. So the probability of an even number from the sum is $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$.
So the probability of an even number from the sum given that the product is even is $\frac{\frac{1}{5}}{\frac{8}{10}} = \frac{1}{5} \times \frac{10}{8} = \frac{1}{4}$.
So the answer is **C**

12. If someone stands on an escalator, it takes 50 seconds to get from one floor to the next. If the escalator is not working, it takes Coco 75 seconds to walk up the same escalator. How long will it take Coco if she walks up the escalator while it is working?

(A) 15 seconds (B) 25 seconds (C) 30 seconds (D) 40 seconds (E) 125 seconds

Solution: We shall use the formula for speed, which is speed = distance divided by the time taken to travel that distance. Let D be the distance of the escalator. Then the speed of the working escalator is $S_e = \frac{D}{50}$. The speed of Coco is $S_c = \frac{D}{75}$. So the speed of Coco walking up the working escalator is $S = S_e + S_c$. But $S = \frac{D}{T}$ and $S_e = \frac{D}{50}$ and $S_c = \frac{D}{75}$. Thus we get;

$$\begin{aligned}
S_e + S_c &= S \\
S_e + S_c &= \frac{D}{T} \\
T &= \frac{D}{S_e + S_c} \\
T &= \frac{D}{(\frac{D}{50} + \frac{D}{75})} \\
T &= \frac{D}{(\frac{3D+2D}{150})} \\
T &= D \div \frac{5D}{150} \\
T &= D \times \frac{30}{D} \\
T &= 30
\end{aligned}$$

Thus it will take Coco 30 seconds to walk up the working escalator.
So the answer is **C**

13. The students in Mr. Wilson's class want to know how old he is. Mr Wilson told them: "My age can be written as the sum of consecutive odd numbers starting from 1 (i.e. $1 + 3 + 5 + \dots$) and is also a multiple of 7". If Mr. Wilson is younger than 100 years old, how old is he?

    (A) 9 (B) 28 (C) 42 (D) 49 (E) 77

    Solution: Lets sum the first 3 odd numbers since the fourth odd number is 7. $1+3+5 = 9$, 9 is not a multiple of 7 so adding 7 to it will not give us a multiple of 7.
    Now let us keep add the next odd numbers except 7 to our sum and see if we get a multiple of 7. $9 + 9 = 18$ so we add 11, $18 + 11 = 29$. So we add 13, $29 + 13 = 42$ which is a multiple of 7.
    To get our answer we just need to add back our 7 we left out so $42 + 7 = 49$. So the answer is **D**

14. How many natural numbers are there with cubes between 101 and 6000?

    (A) 9 (B) 12 (C) 14 (D) 18 (E) 97

    Solution: First let us find the closest cube greater than 101. We know $5^3 = 125$. Now lets look at the upper bound of 6000. We know that $10^3 = 1000$ and $20^3 = 8000$ so the closest cube less than 6000 must be between 10 and 20. Since 8000 is much closer to 6000 than 1000 lets work from 20. So $19^3 = 6859 > 6000$ then $18^3 = 5832 < 6000$. So there are $18 - 4 = 14$ cubes between 101 and 6000.
    So the answer is **C**

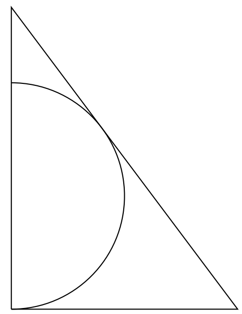
15. The sum of six numbers, a, b, c, d, e and f is 2023. When written in this order, the value of each number differs from the numbers either side by 1. Which of these numbers can possibly equal 338?

(A) All 6 (B) d and f (C) a and f (D) b and d (E) d and e

Solution: Lets take $338 \times 6 = 2028 = 2023 + 5$ so the number in the sum are $338, 338, 337, 337, 337, 336$. If we make $a = 338$ then we can also get that either $c = 338$ or $e = 338$. Now if we make $a = 337$ then we can get that $b = 338$ and also that either $d = 338$ or $f = 338$. So all 6 could be 338.
So the answer is **A**

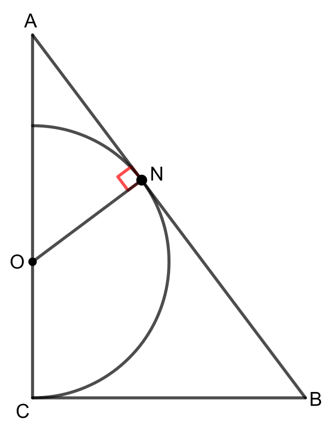
D. 6 point questions

16. A semi-circle is inscribed in a right-angled triangle with base 3 cm and height 4 cm. Determine the radius of the semi-circle.



(A) $\frac{\sqrt{2}}{3}$ (B) $\frac{3}{2}$ (C) $\frac{4\sqrt{5}}{3}$ (D) 2 (E) $\frac{3\sqrt{6}}{4}$

Solution: Lets draw a more detailed diagram.



First we know that the hypotenuse is 5 cm since $3-4-5$ is a pythagorian triple. Next we use the two tangent theorem to get that $CB = NB$ and so $NB = 3$ and $AN = 2$. We also know that $O\widehat{N}A$ is $90°$ by the tangent-radius theorem. Also note that $ON = OC = r$ where r is the radius. So $AO = 4 - r$. Now we can use Pythagorus' theorem to solve for r;

$$\begin{aligned}(4-r)^2 &= r^2 + 2^2 \\ 16 - 8r + r^2 &= r^2 + 4 \\ 12 &= 8r \\ r &= \frac{3}{2}\end{aligned}$$

So the answer is **B**

17. PQR_n represents a 3 digit number in base n. Hence $PQR_n = Pn^2 + Qn + R$ and $P, Q, R \in \{0, 1, 2, ..., n-1\}$. If $ABC_7 = CBA_9$ find C.

A) 1 (B) 2 (C) 3 (D) 5 (E) 6

Solution: From ABC_7 we have that $ABC_7 = A(7)^2 + B(7) + C$ and $A, B, C \in \{0, 1, 2, ..., 6\}$.
From CBA_9 we have that $CBA_9 = C(9)^2 + B(9) + A$ and $A, B, C \in \{0, 1, 2, ..., 8\}$. From $ABC_7 = CBA_9$ we get that $A, B, C \in \{0, 1, 2, ..., 6\}$ and;

$$A(7)^2 + B(7) + C = C(9)^2 + B(9) + A$$
$$49A + 7B + C = 81C + 9B + A$$
$$48A = 80C + 2B$$
$$24A = 40C + B$$

Obviously A and C cannot be 0 because then $A = B = C = 0$ and so we no longer have a 3 digit number. Let us look at the first six multiples of 40; $40, 80, 120, 160, 200, 240$. Lets look at the multiples of 24; $24, 48, 72, 96, 120, 144$ Comparing the multiples we see that the only multiple of 40 that we can add a number in $\{0, 1, 2, ..., 6\}$ and get a multiple of 24 is 120. So that makes $C = 3$.
So the answer is **C**

18. If a, b and c are distinct natural numbers and $\dfrac{6^a 15^b}{9^b 10^c}$ is an integer. Find the order of a, b and c.

(A) $a < b < c$ (B) $b < a < c$ (C) $b < c < a$ (D) $c < a < b$ (E) $c < b < a$

Solution: If $\dfrac{6^a 15^b}{9^b 10^c}$ is a integer then the denominator must 'disappear'. Let us break $6, 15, 9$ and 10 into their prime factors. This gives us; $6 = 2 \times 3$, $15 = 3 \times 5$, $9 = 3 \times 3$ and $10 = 2 \times 5$.
Let us look at the 5, 5 is only in 15 and 10 so to 'get rid' of the 5 in the denominator there must be more 5's in the numerator. This means that the b of the 15^b must be larger than the c of the 10^c. So we have that $c < b$.
Now let us look at the 3. 3 is in both the 6 and 15 in the numerator but only in the 9 in the denominator. However the b in 9^b and 15^b are the same so we won't see a difference the 3's there. So there must be more 3's in the numerator and we look to the 6^a to do that. This means that the a of the 6^a must be larger than the b of the 9^b. So we have that $b < a$.
Putting these together we get that $c < b < a$.
So the answer is **E**

19. Find the number of solutions to the equation

$$|2x - 3| = \lfloor x \rfloor.$$

Note. $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x. *For example* $\lfloor 3.1 \rfloor = 3$ *and* $\lfloor -6.8 \rfloor = -7$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: First since $|2x-3|$ will always give us a positive number we only have to look at $x \geq 0$. Lets break it down into 2 cases.
Case 1 ($0 \leq x < \frac{3}{2}$): Here $2x-3 < 0$ so $|2x-3| = -(2x-3) = 3-2x$. Here $\lfloor x \rfloor = 0$ or 1. If $\lfloor x \rfloor = 0$ then $3-2x = 0$ and so $x = \frac{3}{2}$ but $\lfloor \frac{3}{2} \rfloor = 1$ so this is not a solution. If $\lfloor x \rfloor = 1$ then $3-2x = 1$ and so $x = 1$ and $\lfloor 1 \rfloor = 1$ so this is a solution.
Case 2 ($x \geq \frac{3}{2}$): Here $2x-3 \geq 0$ so $|2x-3| = 2x-3$. Now let $n = \lfloor x \rfloor$ then $n \leq x < n+1$. We get that $2x-3 = n$ and so $x = \frac{n+3}{2}$. We substitute this value of x into $n \leq x < n+1$;

$$n \leq \frac{n+3}{2} < n+1$$
$$2n \leq n+3 < 2n+2$$

Looking at $2n \geq n+3$ we get that $n \leq 3$. Looking at $n+3 < 2n+2$ we get that $n > 1$. Putting this together we get that $n = 2$ or 3.
If $\lfloor x \rfloor = 2$ then $2x-3 = 2$ and so $x = \frac{5}{2}$ and $\lfloor \frac{5}{2} \rfloor = 2$ so this is a solution. If $\lfloor x \rfloor = 3$ then $2x-3 = 3$ and so $x = 3$ and $\lfloor 3 \rfloor = 3$ so this is a solution.
Thus we have 3 solutions.
So the answer is **C**

20. Let $x_1, \ldots, x_n$ be a list of n distinct positive integers (each $x_i \geq 1$) such that $x_i + x_j + x_k$ is prime whenever i, j, k are distinct. What is the maximum value of n?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: D