2023 Wits Mathematics Competition
Qualifying Round
Junior Secondary

Instructions

This exam consists of 20 multiple choice questions. There is one correct answer to each question. There is no penalty for incorrect answers. The mark allocation is as follows:

Questions 1-5 are each worth 3 points,
Questions 6-10 are each worth 4 points,
Questions 11-15 are each worth 5 points,
Questions 16-20 are each worth 6 points.
The total number of points available is 90.

The time limit on this exam is 75 minutes, calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed. If you are using the computer-friendly answer sheet you should fill it in in BLACK pen (other colours do not scan well). Time may be given for filling in name, school and other personal details.

"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?". — David Hilbert

A. 3 point questions

1. Compute $0,3202 - 0,2023$.

(A) $0,0179$ (B)$0,0979$ (C) $0,1079$ (D) $0,1179$ (E) $0,1279$

Solution: $0,3000-0,2000 = 0,1000$ and $0,0202-0,0023 = 0,0179$ so $0,3202-0,2023 = 0,1000 + 0,0179 = 0,1179$.
So the answer is **D**

2. A cake weighing 1200g is cut into five pieces. The largest slice weighs half as much as the other four together. Find the weight of the largest slice.

(A) 200g (B) 300g (C) 400g (D) 600g (E) 800g

Solution: Since the largest slice weighs half as much as the other 4 together it weighs as much as 2 of the other slices. So we count the largest slice as 2 and then divide 1200g by 6 to get the weight of the normal slices. $1200 \div 6 = 200$, so a normal slice weighs 200g and since the largest slice is twice the weight of a normal slice it weighs, $2 \times 200 = 400$g.
So the answer is **C**

3. Thabo had the 70^{th} best score in the WMC at his school and the 33^{rd} worst score. How many people at his school took part in the WMC?

(A) 99 (B) 100 (C) 101 (D) 102 (E) 103

Solution: Since Thabo had the 70^{th} best score that means there are 69 people in his school that did better than him. Furthermore, since he had the 33^{rd} worst score that means that 32 people did worse than him. So the total number of people who took part is the number of people who did better than him, the number of people who did worse than him and then him. So $69 + 32 + 1 = 102$ people.
So the answer is **D**

4. Sandile has completed $\frac{3}{7}$ of her Mathematics homework. If she still has to solve 12 problems, how many problems did she receive as homework altogether?

(A) 12 (B) 14 (C) 21 (D) 35 (E) 42

Solution: Since she already completed $\frac{3}{7}$ that means she has $\frac{4}{7}$ of her problems left to do. Let x be the total number of problems. So;

$$\begin{aligned} \frac{4}{7} \times x &= 12 \\ 4 \times x &= 7 \times 12 \\ x &= 84 \div 4 \\ x &= 21 \end{aligned}$$

So the answer is **C**

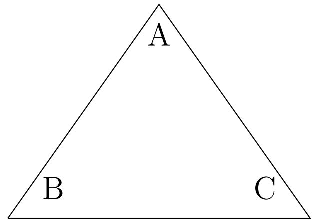
5. Which of the following rational numbers does not lie between $\frac{2}{5}$ and $\frac{3}{4}$?

   (A) $\frac{27}{60}$ (B) $\frac{1}{2}$ (C) $\frac{5}{13}$ (D) $\frac{7}{12}$ (E) $\frac{3}{5}$

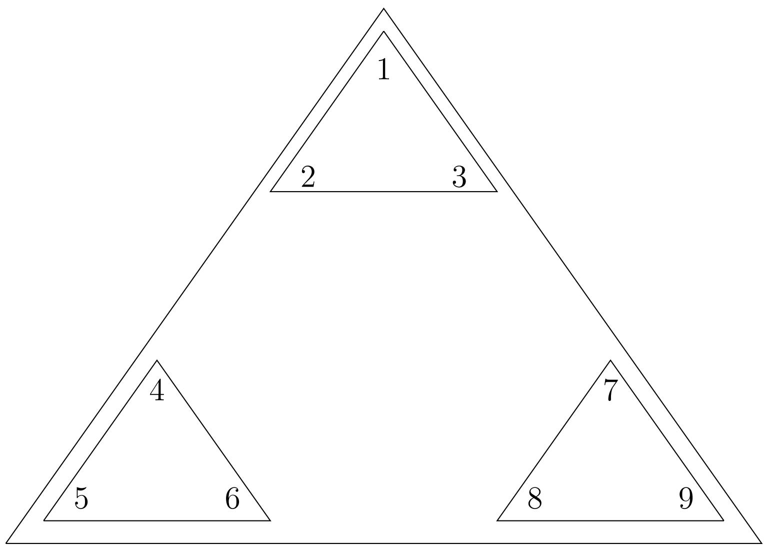
   Solution: First we get a common denominators, $\frac{2}{5} = \frac{2\times4}{5\times4} = \frac{8}{20}$ and $\frac{3}{4} = \frac{3\times5}{4\times5} = \frac{15}{20}$. We can immediately see that $\frac{1}{2}(= \frac{10}{20})$ and $\frac{3}{5}(= \frac{12}{20})$ are between our two fractions. Now for $\frac{7}{12}$, it is above $\frac{1}{2}$ so we just need to check it is not above $\frac{3}{4}$. Since $\frac{3}{4} = \frac{9}{12} > \frac{7}{12}$, it is between our 2 fractions. For $\frac{27}{60}$, it is less than $\frac{1}{2}$ so we need to check it is not below $\frac{2}{5}$. Since $\frac{2}{5} = \frac{24}{60}$, so $\frac{27}{60}$ lies between our 2 fractions. Thus $\frac{5}{13}$ does not.
   So the answer is **C**

B. 4 point questions

6. The value of the figure below is given by $\frac{A+B}{C}$.

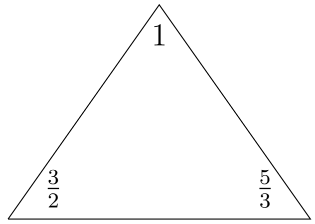


Find the value of:



(A) $\frac{2}{3}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) $\frac{21}{8}$

Solution: For the top triangle; $\frac{1+2}{3} = \frac{3}{3} = 1$. For the bottom left triangle; $\frac{4+5}{6} = \frac{9}{6} = \frac{3}{2}$. For the bottom right triangle; $\frac{7+8}{9} = \frac{15}{9} = \frac{5}{3}$. Now the big triangle looks as follows;



So now we get; $\frac{1+\frac{3}{2}}{\frac{5}{3}} = (\frac{2}{2} + \frac{3}{2}) \div \frac{5}{3} = \frac{5}{2} \times \frac{3}{5} = \frac{3}{2}$.
So the answer is **C**

7. 12 students are evenly spaced along the circumference of a circle. The students are numbered from 1 to 12. Which student sits opposite student 8?

(A) Student 2 (B) Student 3 (C) Student 4 (D) Student 10 (E) Student 12

Solution: 12 students are evenly spaced along the circumference of a circle ends up looking exactly like a clock. So in a clock the number opposite 8 is 2.
So the answer is **A**

8. The final score at a soccer match was Pirates m and Chiefs n. How many different scores were possible at half-time?

(A) $m+n$ (B) $n \times m$ (C) $2 \times m \times n$ (D) $(m-1) \times (n-1)$
(E) $(m+1) \times (n+1)$

Solution: At half-time each team could score any amount of goals between 0 and the amount of goals they have at the end of the match.

Pirates' Goals	Chiefs' Goals
0	0, 1, 2, ..., n
1	0, 1, 2, ..., n
2	0, 1, 2, ..., n
⋮	⋮
m	0, 1, 2, ..., n

From the table we see that for each possible number of goals Pirates has there are $n+1$ possible number of Chiefs' goals. Thus there are $m+1$ lots of $n+1$ so the total different possible scores at half-time is $(m+1) \times (n+1)$.
So the answer is **E**

9. $2 \times 2 \times 2 \times ... \times 2$ (40 '2' symbols) can be written as 2^{40}. What is half of 2^{40}?

(A) 2^{20} (B) 2^{30} (C) 2^{35} (D) 2^{39} (E) 2^{40}

Solution: When we take half of a number we divide it by 2, so then in $2 \times 2 \times 2 \times ... \times 2$ there is one less 2 than the 40 there originally was. So, there is 39 2's, i.e. 2^{39}.
So the answer is **D**

10. A, B and C are different chemicals. One part of A is mixed with three parts of B to create a 10ml mixture, called U. Two parts of B is mixed with three parts of C to create a 10ml mixture, called V. Finally, one part of U is mixed with two parts of V to create a 10ml mixture, called W. What is the ratio of chemical A to B in the final mixture W?

(A) 1 : 2 (B) 6 : 37 (C) 5 : 29 (D) 1 : 7 (E) 5 : 31

Solution: Let us turn our ratios into fractions.
Mixture U is made of one part of A and three parts of B, so in U there is $10 \times \frac{1}{4} = \frac{10}{4}$ml of A and $10 \times \frac{3}{4} = \frac{30}{4}$ml of B. To get the amount of A and B per ml of U we divide the total mls of A and B in U by 10. Thus in 1ml of U there is $\frac{1}{4}$ml of A and $\frac{3}{4}$ml of B.

Mixture V is made of two parts of B and three parts of C, so in V there is $10 \times \frac{2}{5} = \frac{20}{5}$ml of B and $10 \times \frac{3}{5} = \frac{30}{5}$ml of C. To get the amount of B and C per ml of V we divide the total mls of B and C in V by 10. Thus in 1ml of V there is $\frac{2}{5}$ml of B and $\frac{3}{5}$ml of C.
Mixture W is made of one part of U and two parts of V, so there is $10 \times \frac{1}{3} = \frac{10}{3}$ml of U and $10 \times \frac{2}{3} = \frac{20}{3}$ml of V. Now we want to know how many mls are A, B and C.
The amount of C in W is $\frac{3}{5}$ of $\frac{20}{3}$ (the amount of V in W). So we have $\frac{3}{5} \times \frac{20}{5} = 4$ml of C. This also tells us that there is 6ml of A and B in W.
The amount of A in W is $\frac{1}{4}$ of $\frac{10}{3}$ (the amount of U in W). So we have $\frac{1}{4} \times \frac{10}{3} = \frac{5}{6}$ml of A. The amount of B in W is $\frac{3}{4}$ of $\frac{10}{3}$ (the amount of U in W) AND $\frac{2}{5}$ of $\frac{20}{3}$ (the amount of V in W). So we have $\frac{2}{3} \times \frac{10}{3} + \frac{2}{5} \times \frac{20}{3} = \frac{5}{2} + \frac{8}{3} = \frac{31}{6}$ml of B.
Thus our ratio of A to B is $5 : 31$.
So the answer is **E**

C. 5 point questions

11. The four-digit number $3XX1$ is divisible by 9. What digit does X represent?

(A) 0 (B) 2 (C) 5 (D) 7 (E) 9

Solution: We can use a process of elimination here.
If $X = 0$ then our four-digit number is 3001 and $3001 \div 9 = 333$ remainder 4.
If $X = 2$ then our four-digit number is 3221 and $3221 \div 9 = 357$ remainder 8.
If $X = 5$ then our four-digit number is 3551 and $3551 \div 9 = 394$ remainder 5.
If $X = 7$ then our four-digit number is 3771 and $3771 \div 9 = 419$. So, $X = 7$.
So the answer is **D**

12. A painter takes two days to paint a room (all four walls and the ceiling). If he works at the same pace, how many days will he take to paint a room that is twice as wide, twice as long, and twice as high?

(A) 4 (B) 8 (C) 10 (D) 12 (E) 16

Solution: The original room has 5 surfaces that are painted which are rectangles. If we double the length and height of the original wall to make a new wall then the new wall will be 4 times the size or 4 rectangles. But we have 4 walls so in the new room we have 16 rectangles ($4 \times 4 = 16$). If we do the same with ceiling, doubling the length and width, the ceiling is 4 times bigger or 4 rectangles. There is only one ceiling. So the total rectangles the painter will need to paint in the new room is $16 + 4 = 20$ rectangles. Since $20 \div 5 = 4$ there are 4 times as many original surfaces to paint, so if it took 2 days to paint the original 5 surfaces it will take $4 \times 2 = 8$ days to paint the 20 of the original surfaces.
So the answer is **B**

13. The areas of three adjacent faces of a rectangular box are 6, 8, and 12. What is the volume of this rectangular box?

(A) 24 (B) 26 (C) 28 (D) 32 (E) 48

Solution: First we look at the factors of 6, 8 and 12. For 6; 1, 2, 3, 6. For 8; 1, 2, 4, 8. For 12; 1, 2, 3, 4, 6, 12. So, lengths of the sides of the face with an area of 6 are 2 and 3. The side of length 2 is on the face of area 8 and the side of length 3 is on the face of area 12. The volume of a rectangular box is length $\times$ width $\times$ height which can be simplified to just be the area of one face $\times$ the length of the side not on the face. So we get that the volume of our rectangular box is $3 \times 8 = 24$.
So the answer is **A**

14. Let $x_1, \ldots, x_n$ be a list of n distinct positive integers (each $x_i \geq 1$) such that $x_i + x_j$ is prime whenever $i \neq j$. What is the maximum value of n?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Besides 2 all prime numbers are odd numbers. To get an odd number from a sum of two numbers we need to add an odd number and an even number. Let x_1 be an odd number and x_2 be an even number. Then if we want to make another prime from $x_1 + x_3$ then x_3 needs to be an even number but then $x_2 + x_3$ is an even number and so not a prime number. So we can only have 2 numbers in our list ($n = 2$).
So the answer is **B**

15. A farmer has 40 cows and some chickens. The total number of legs of the chickens is equal to twice the total number of legs of the cows. How many chickens does the farmer have? (Note: Cows have four legs and chickens have two legs.)

(A) 20 (B) 40 (C) 80 (D) 120 (E) 160

Solution: Every cow has 4 legs so the number of legs of cows is $40 \times 4 = 160$. We know that the number of legs of chickens is twice the number of legs of cows which gives us $2 \times 160 = 320$ legs of chickens. Every chicken has 2 legs so we divide the total number of legs of chickens by 2 to get the number of chickens. So there are $320 \div 2 = 160$ chickens.
So the answer is **E**

D. 6 point questions

16. If someone stands on an escalator, it takes 50 seconds to get from one floor to the next. If the escalator is not working, it takes Coco 75 seconds to walk up the same escalator. How long will it take Coco if she walks up the escalator while it is working?

(A) 15 seconds (B) 25 seconds (C) 30 seconds (D) 40 seconds (E) 125 seconds

Solution: We shall use the formula for speed, which is speed = distance divided by the time taken to travel that distance. Let D be the distance of the escalator. Then the speed of the working escalator is $S_e = \frac{D}{50}$. The speed of Coco is $S_c = \frac{D}{75}$. So the speed of Coco walking up the working escalator is $S = S_e + S_c$. But $S = \frac{D}{T}$ and $S_e = \frac{D}{50}$ and $S_c = \frac{D}{75}$. Thus we get;

$$
\begin{aligned}
S_e + S_c &= S \\
S_e + S_c &= \frac{D}{T} \\
T &= \frac{D}{S_e + S_c} \\
T &= \frac{D}{\left(\frac{D}{50} + \frac{D}{75}\right)} \\
T &= \frac{D}{\left(\frac{3D+2D}{150}\right)} \\
T &= D \div \frac{5D}{150} \\
T &= D \times \frac{30}{D} \\
T &= 30
\end{aligned}
$$

Thus it will take Coco 30 seconds to walk up the working escalator.
So the answer is **C**

17. Find the last digit of $2023^{(2023^{2023})}$.

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Solution: To solve this we need to find a pattern in the last digits.
Obviously the last digit of 2023^1 is **3**.
For the last digit of 2023^2 we multiply the last digit of 2023. So we get $3 \times 3 = \mathbf{9}$.
For 2023^3 we multiply the last digit of 2023^2 by the last digit of 2023. So we get $9 \times 3 = 27$, making the last digit **7**.
For 2023^4 we multiply the last digit of 2023^3 by the last digit of 2023. So we get $7 \times 3 = 21$, making the last digit **1**.
For 2023^5 we multiply the last digit of 2023^4 by the last digit of 2023. So we get $1 \times 3 = 3$, making the last digit **3**.

So we can see that the last digit is in a pattern of size 4. Thus to get the last digit of 2023^{2023} we divide 2023 by 4 and the remainder will tell us where in the pattern we are. If the remainder is 1 then the last digit is 3, if it's 2 then the last digit is 9, if it's 3 then the last digit is 7 and if there is no remainder then the last digit is 1.
So $2023 \div 4$ gives us a remainder of 3 so the last digit of 2023^{2023} is 7. Now to get the last digit of $2023^{2023^{2023}}$ we divide the last digit of 2023^{2023} by 4 and again see where we sit in our pattern. Since the last digit of 2023^{2023} is 7 we divide 7 by 4 which again gives us a remainder of 3. Thus our last digit of $2023^{2023^{2023}}$ is 7. So the answer is **D**

18. If a,b and c are distinct natural numbers (positive integers) and $\frac{6^a 15^b}{9^b 10^c}$ is an integer. Find the order of a, b and c.

(A) $a < b < c$ (B) $b < a < c$ (C) $b < c < a$ (D) $c < a < b$ (E) $c < b < a$

Solution: If $\frac{6^a 15^b}{9^b 10^c}$ is a integer then the denominator must 'disappear'. Let us break $6, 15, 9$ and 10 into their prime factors. This gives us; $6 = 2 \times 3$, $15 = 3 \times 5$, $9 = 3 \times 3$ and $10 = 2 \times 5$.
Let us look at the 5, 5 is only in 15 and 10 so to 'get rid' of the 5 in the denominator there must be more 5's in the numerator. This means that the b of the 15^b must be larger than the c of the 10^c. So we have that $c < b$.
Now let us look at the 3. 3 is in both the 6 and 15 in the numerator but only in the 9 in the denominator. However the b in 9^b and 15^b are the same so we won't see a difference the 3's there. So there must be more 3's in the numerator and we look to the 6^a to do that. This means that the a of the 6^a must be larger than the b of the 9^b. So we have that $b < a$.
Putting these together we get that $c < b < a$.
So the answer is **E**

19. What is the sum of the last two digits of the product $1 \times 2 \times 3 \times 4 \times 5 \times 4 \times 3 \times 2 \times 1$

(A) 0 (B) 4 (C) 8 (D) 12 (E) 16

Solution: Lets simplify the product. First multiplying anything by 1 just gives us the same thing so we can just drop the 1's. Next $3 \times 4 = 12$ and there are 2 3×4 so we get $12 \times 12 = 144$. Next $2 \times 5 = 10$ so $144 \times 10 = 1440$. Finally we have one 2 left and $1440 \times 2 = 2880$. So the last 2 digits are 8 and 0 and $8 + 0 = 8$.
So the answer is **C**

20. PQR_n represents a 3 digit number in base n. Hence $PQR_n = Pn^2 + Qn + R$ and $P, Q, R \in \{0, 1, 2, ..., n-1\}$. If $ABC_7 = CBA_9$ find C.

A) 1 (B) 2 (C) 3 (D) 5 (E) 6

Solution: From ABC_7 we have that $ABC_7 = A(7)^2 + B(7) + C$ and $A, B, C \in \{0, 1, 2, ..., 6\}$.
From CBA_9 we have that $CBA_9 = C(9)^2 + B(9) + A$ and $A, B, C \in \{0, 1, 2, ..., 8\}$. From $ABC_7 = CBA_9$ we get that $A, B, C \in \{0, 1, 2, ..., 6\}$ and;

$$\begin{aligned} A(7)^2 + B(7) + C &= C(9)^2 + B(9) + A \\ 49A + 7B + C &= 81C + 9B + A \\ 48A &= 80C + 2B \\ 24A &= 40C + B \end{aligned}$$

Obviously A and C cannot be 0 because then $A = B = C = 0$ and so we no longer have a 3 digit number. Let us look at the first six multiples of 40; $40, 80, 120, 160, 200, 240$. Lets look at the multiples of 24; $24, 48, 72, 96, 120, 144$ Comparing the multiples we see that the only multiple of 40 that we can add a number in $\{0, 1, 2, ..., 6\}$ and get a multiple of 24 is 120. So that makes $C = 3$.
So the answer is **C**