2022 Wits Mathematics Competition
Qualifying Round
Undergraduate

Instructions

This exam consists of 20 multiple choice questions. There is one correct answer to each question. There is no penalty for incorrect answers. The mark allocation is as follows:

Questions 1-5 are each worth 3 points,
Questions 6-10 are each worth 4 points,
Questions 11-15 are each worth 5 points,
Questions 16-20 are each worth 6 points.
The total number of points available is 90.

The time limit on this exam is 75 minutes, calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed. If you are using the computer-friendly answer sheet you should fill it in in BLACK pen (other colours do not scan well). Time may be given for filling in name, school and other personal details.

It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profoundity, he is talking nonsense" — Alfred North Whitehead

A. 3 point questions

1. Compute $2022 - 1234$.

   A) 777 B) 788 C) 877 D) 886 E) 920

   Solution B: This can be done by subtraction or simply by looking at the last digit of the available options.

2. A water tank is $\frac{5}{6}$ full. When 30 litres is released the tank is $\frac{4}{5}$ full. Find the capacity of the tank in litres.

   A) 275 B) 300 C) 900 D) 1200 E) 1500

   Solution: C. Let the capacity be x then $\frac{5x}{6} - \frac{4x}{5} = 30$, which is easily solved to $x = 900$.

3. The product of two positive integers is equal to twice their sum. The same product is also equal to 6 times the difference between the two integers. What is the sum of the integers?

   A) 6 B) 9 C) 12 D) 15 E) 16

   Solution: B. Call the smaller number a and the larger one b. Then $2a + 2b = 6b - 6a$ which solves to $b = 2a$. Substituting into $ab = 2a + 2b$ gives $2a^2 = 6a$ and $a = 3$, $b = 6$. These sum to 9.

4. For a three-digit number xyz, where x, y and z represent the digits in base 10, the function $h(xyz) = 5^x 2^y 3^z$. If $h(abc) = 3h(xyz)$ find $abc - xyz$.

   A) 1 B) 3 C) 10 D) 150 E) 280

   Solution: A. Plugging things in gives $5^a 2^b 3^c = 3 \times 5^x 2^y 3^z = 5^x 2^y 3^{z+1}$ to give $a = x$, $b = y$ and $c = z + 1$.

5. The letters of the alphabet are numbered consecutively, starting from an arbitrary number. We have that

   $$F + I + I = 2022$$

   what is the average of X, Y and Z?

   A) 123 B) 689 C) 691 D) 696 E) 701

   $F = A+5$ and $I = A+8$. Then, $A+5+2(A+8) = 2022$ Hence $A = (2022-21)/3 = 667$. Then, the average of X, Y and Z is just $Y (X = Y-1, Z = Y+1)$. So, $Y = A+24 = 691$.

B. 4 point questions

6. Evaluate:

$$\int_{-1}^{1} |(x-1)x(x+1)|\,dx$$

A) 0 B) $\frac{1}{2}$ C) 1 D) $\frac{4}{3}$ E) 2

Solution: B

$$\begin{aligned}\int_{-1}^{1} |(x-1)x(x+1)|\,dx &= \int_{-1}^{0} x^3 - x\,dx + \int_{0}^{1} x - x^3\,dx \\ &= 2\int_{0}^{1} x^3 - x\,dx \\ &= 2(\frac{1}{2} - \frac{1}{4}) \\ &= \frac{1}{2}\end{aligned}$$

7. Evaluate:

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(x)}$$

A) 0 B) 1 C) 3 D) 12 E) ∞

Solution: C. Use the Taylor series of L'hopital's rule

8. Find the sum of all values of x satisfying the following simultaneous equations:

$$\begin{aligned} x^2 + 3y &= 10, \\ 3 + y &= \frac{10}{x}. \end{aligned}$$

A) -3 B) -2 C) 0 D) 4 E) 5

Solution: C. Note that we can write $y = 3 - \frac{10}{x}$. Substituting this into the other equation gives $x^3 + 30 - 9x = 10x$ or $x^3 - 19x + 30 = 0$. Which may be factorized as $(x-2)(x-3)(x+5) = 0$ giving roots of $x = -2, -3$ and 5.

9. If $N = \frac{33!}{22!}$ find the largest integer k such that N is a multiple of 6^k.

A) 4 B) 6 C) 8 D) 11 E) 15

Answer: B 6 Solution: We write out the numbers 23 through 33 and count six factors of three and an abundance of factors of twos.

10. Three boys are aged 4, 6 and 7. Three girls are aged 5, 8 and 9. Two boys and two girls are selected randomly. Find the probability that the sum of the selected children's ages is even.

A) 0 B) $\frac{2}{9}$ C) $\frac{4}{9}$ D) $\frac{5}{9}$ E) 1

Solution: D $\frac{5}{9}$. Notice that the sum of all six children's ages is odd (as three of them are of odd ages. To get an even sum the ages of the two children not picked must add up to an odd number. Either an even-aged boy and an odd-aged girl or an odd aged-girl and even aged boy. There are five ways to do this and nine ways to choose the two left out children.

C. 5 point questions

11. A regular polygon with 6 sides has 9 diagonals. How many diagonals does a regular polygon with 1000 sides have?

A) 495800 B) 496200 C) 497000 D) 498000 E) 498500

Solution E: There are $\binom{1000}{2} = 499500$ ways to choose two vertices. However we must subtract the 1000 non-diagonal lines which represent the sides to get 498500.

12. Find the ten-digit of the two smallest positive integers with exactly 10 distinct factors.

A) 48 B) 128 C) 210 D) 512 E) 560

Answer: B 128 Solution: Our options for numbers with ten factors are those numbers of the form p^9 or pq^4 where p and q are distinct primes for which we wish to minimise. Try 2^9 and 3×2^4 and we get 48 is the smallest and $80 = 5 \times 2^4$.

13. How many positive integers below one thousand contain exactly three ones when written in binary? Equivalently how many positive integers below one thousand can be written as a sum of three different powers of two.

A) 90 B) 120 C) 150 D) 180 E) 210

Solution: B 120. The question can be reduced to choosing three positions for the ones in a ten digit binary expansion with leading zeros allowed. There are $\binom{10}{3} = 120$ ways to do this.

14. The function $f(x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + ...$ Find the derivative of $f(x)$ at $x = \frac{1}{2}$.

A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) -2 D) $\frac{2}{3}$ E) -1

Solution A. $f'(x) = x - x^2 + x^3 - ...$ which is a geometric series.

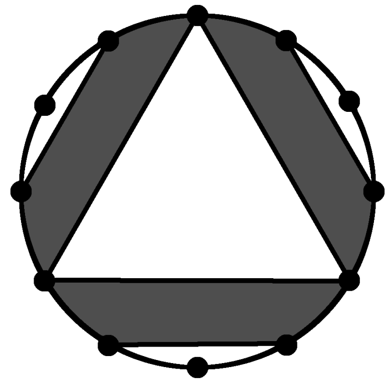
15. Let f and g be continuous functions such that $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = 1$. Find the smallest c such that it is guaranteed that $f(x) + g(x) \geq c$ for some $x \in (0, 1)$.

A) $\frac{1+\sqrt{5}}{2}$ B) $e - 1$ C) 2 D) 3 E) π

Solution: C. The average value of $f + g$ in $(0, 1)$ is 2 hence the maximum must be at least 2 and 2 can be achieved by using the constant functions $f(x) = g(x) = 1$.

D. 6 point questions

16. In the diagram below the circle has radius 6 and the dots are equally spaces. Find the value of the shaded area.



A) $36\pi - 27\sqrt{3}$ B) 15π C) 50 D) 18π E) $27(\pi - \sqrt{3} + 1)$

Solution: D. First note that the entire circle has area 36π. Our strategy will be to compute the size of the unshaded area and subtract it from this. First we shall compute the area of the three identical small unshaded regions which lie 'beyond' the shaded region. We compute that these have area $6\pi - 9\frac{\sqrt{3}}{2}$. Too see this label the two marked points at the boundary of this unshaded triangle A and B and the big circle's center O. These form an equilateral triangle with side length 6. The unshaded arc's area is given by subtracting the area of this triangle by the sixth part of the circle with spanned by the arc AB. We then consider the central triangle. A little angle chasing shows that it has side length $6\sqrt{3}$ and area of $\frac{27\sqrt{3}}{2}$.

This gives a total shaded area of $36\pi - 3(6\pi - \frac{9\sqrt{3}}{2}) - \frac{27\sqrt{3}}{2} = 18\pi$

17. Find the range of values of k for which the equation $3x^4 + 4x^3 - 12x^2 + k = 0$ has 4 real roots.

A) $0 < k < 10$ B) $-1 < k < 1$ C) $0 < k < 5$ D) $-1 < k < 5$ E) $5 < k < 10$

C. Graph the function $g(x) = 3x^4 + 4x^3 - 12x^2$ and consider where $g(x) = -k$ has 4 solutions.

18. In the Wits Mathematical Lottery, exactly N tickets are sold. There is a total prize pool of C Wits Mathematical Tokens. Each ticket has a probability p of being a winning ticket, independently of each of the other tickets. The prize pool is divided between all of the winning tickets, unless there are no winning tickets in which case the prize is not awarded. Everyone else gets nothing. What is the expected value of a ticket? That is what is the average amount you'd win on a particular ticket?

A) 0 B) $\frac{C}{N}(1+p^N)$ C) $\frac{C}{N}$ D) $\frac{C}{N}[1-(1-p)^N]$ E) $\frac{pC}{N}$

Solution: A given ticket has a probability p of being a winning ticket. The probability that there are k other winning tickets is $\binom{N-1}{k}p^k(1-p)^{N-1-k}$, and in this case the payout for the winning ticket is $\frac{C}{k+1}$. So the expected value is $p\sum_{k=0}^{N-1}\binom{N-1}{k}p^k(1-p)^{N-1-k}\frac{C}{k+1}$. Using the identity $\binom{N}{k+1} = \frac{N}{k+1}\frac{N-1}{k}$ this simplifies to $\frac{C}{N}\sum_{k=0}^{N-1}\binom{N}{k+1}p^{k+1}(1-p)^{N-1-k} = \frac{C}{N}[1-(1-p)^N]$

Alternative: The total expected value of all of the tickets is 0 WM Tokens if nobody wins, and C WM Tokens if at least one person wins. The probability that at least one person wins is $(1-(1-p)^N)$. By linearity of expectation, the expected value of each individual ticket is $CN(1-(1-p)^N)$.

19. Let $[n] = \{1, 2, \ldots, n\}$. For T a non-empty subset of $[n]$ let M_T be the reciprocal of the product of elements of T. Define S_n as the sum of M_T across all such non-empty subsets of $[n]$. Find S_{2022}.

A) 1621 B) 1811 C) 2022 D) 4044 E) 6000

Solution: D 2022. More generally it can be shown by induction that $S_n = n$. Once one observes the recurrence $S_{n+1} = S_n + \frac{1}{n+1}S_n + \frac{1}{n+1}$ this becomes straight forward.

20. Evaluate:

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}+1} \cos(\,\arcsin(\,\arccos(\,\sin(x)\,)\,)\,)\,dx$$

A) $\frac{1}{2}$ B) $\frac{\pi}{4}$ C) 1 D) $\frac{\pi}{2}$ E) e

Solution: (D) The expression simplifies to a semi-circle with radius 1 for the given bounds. The answer is thus the area of this semi-circle which is just $\frac{1}{2}\cdot\pi\cdot 1^2 = \frac{\pi}{2}$.