



2022 Wits Mathematics Competition
Qualifying Round
Undergraduate

Instructions

This exam consists of 20 multiple choice questions. There is one correct answer to each question. There is no penalty for incorrect answers. The mark allocation is as follows:

Questions 1-5 are each worth 3 points,
Questions 6-10 are each worth 4 points,
Questions 11-15 are each worth 5 points,
Questions 16-20 are each worth 6 points.
The total number of points available is 90.

The time limit on this exam is 75 minutes, calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed. If you are using the computer-friendly answer sheet you should fill it in in BLACK pen (other colours do not scan well). Time may be given for filling in name, school and other personal details.

It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profundity, he is talking nonsense” — Alfred North Whitehead

A. 3 point questions

1. Compute $2022 - 1234$.

- A) 777 B) 788 C) 877 D) 886 E) 920

2. A water tank is $\frac{5}{6}$ full. When 30 litres is released the tank is $\frac{4}{5}$ full. Find the capacity of the tank in litres.

- A) 275 B) 300 C) 900 D) 1200 E) 1500

3. The product of two positive integers is equal to twice their sum. The same product is also equal to 6 times the difference between the two integers. What is the sum of the integers?

- A) 6 B) 9 C) 12 D) 15 E) 16

4. For a three-digit number xyz , where x , y and z represent the digits in base 10, the function $h(xyz) = 5^x 2^y 3^z$. If $h(abc) = 3h(xyz)$ find $abc - xyz$.

- A) 1 B) 3 C) 10 D) 150 E) 280

5. The letters of the alphabet are numbered consecutively, starting from an arbitrary number. We have that

$$F + I + I = 2022$$

what is the average of X , Y and Z ?

- A) 123 B) 689 C) 691 D) 696 E) 701

B. 4 point questions

6. Evaluate:

$$\int_{-1}^1 |(x-1)x(x+1)| dx$$

- A) 0 B)
- $\frac{1}{2}$
- C) 1 D)
- $\frac{4}{3}$
- E) 2

7. Evaluate:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(x)}$$

- A) 0 B) 1 C) 3 D) 12 E)
- ∞

8. Find the sum of all values of x satisfying the following simultaneous equations:

$$\begin{aligned}x^2 + 3y &= 10, \\3 + y &= \frac{10}{x}.\end{aligned}$$

- A)
- -3
- B)
- -2
- C) 0 D) 4 E) 5

9. If $N = \frac{33!}{22!}$ find the largest integer k such that N is a multiple of 6^k .

- A) 4 B) 6 C) 8 D) 11 E) 15

10. Three boys are aged 4, 6 and 7. Three girls are aged 5, 8 and 9. Two boys and two girls are selected randomly. Find the probability that the sum of the selected children's ages is even.

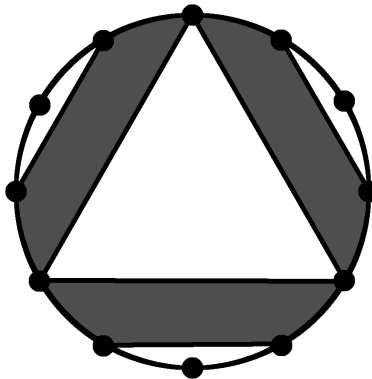
- A) 0 B)
- $\frac{2}{9}$
- C)
- $\frac{4}{9}$
- D)
- $\frac{5}{9}$
- E) 1

C. 5 point questions

11. A regular polygon with 6 sides has 9 diagonals. How many diagonals does a regular polygon with 1000 sides have?
- A) 495800 B) 496200 C) 497000 D) 498000 E) 498500
12. Find the sum of the two smallest positive integers with exactly 10 distinct factors.
- A) 48 B) 128 C) 210 D) 512 E) 560
13. How many positive integers below one thousand contain exactly three ones when written in binary? Equivalently how many positive integers below one thousand can be written as a sum of three different powers of two.
- A) 90 B) 120 C) 150 D) 180 E) 210
14. The function $f(x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \dots$. Find the derivative of $f(x)$ at $x = \frac{1}{2}$.
- A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) -2 D) $\frac{2}{3}$ E) -1
15. Let f and g be continuous functions such that $\int_0^1 f(x) dx = \int_0^1 g(x) dx = 1$. Find the smallest c such that it is guaranteed that $f(x) + g(x) \geq c$ for some $x \in (0, 1)$.
- A) $\frac{1+\sqrt{5}}{2}$ B) $e - 1$ C) 2 D) 3 E) π

D. 6 point questions

16. In the diagram below the circle has radius 6 and the dots are equally spaces. Find the value of the shaded area.



- A) $36\pi - 27\sqrt{3}$ B) 15π C) 50 D) 18π E) $27(\pi - \sqrt{3} + 1)$
17. Find the range of values of k for which the equation $3x^4 + 4x^3 - 12x^2 + k = 0$ has 4 real roots.
- A) $0 < k < 10$ B) $-1 < k < 1$ C) $0 < k < 5$ D) $-1 < k < 5$ E) $5 < k < 10$
18. In the Wits Mathematical Lottery, exactly N tickets are sold. There is a total prize pool of C Wits Mathematical Tokens. Each ticket has a probability p of being a winning ticket, independently of each of the other tickets. The prize pool is divided between all of the winning tickets, unless there are no winning tickets in which case the prize is not awarded. Everyone else gets nothing. What is the expected value of a ticket? That is what is the average amount you'd win on a particular ticket?
- A) 0 B) $\frac{C}{N}(1 + p^N)$ C) $\frac{C}{N}$ D) $\frac{C}{N}[1 - (1 - p)^N]$ E) $\frac{pC}{N}$
19. Let $[n] = \{1, 2, \dots, n\}$. For T a non-empty subset of $[n]$ let M_T be the reciprocal of the product of elements of T . Define S_n as the sum of M_T across all such non-empty subsets of $[n]$. Find S_{2022} .
- A) 1621 B) 1811 C) 2022 D) 4044 E) 6000
20. Evaluate:

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}+1} \cos(\arcsin(\arccos(\sin(x)))) dx$$

- A) $\frac{1}{2}$ B) $\frac{\pi}{4}$ C) 1 D) $\frac{\pi}{2}$ E) e