2022 Wits Mathematics Competition
Final Round
Undergraduate

Instructions

This paper is 90 minutes long and consists of ten single answer questions (to be answered in the below table) and two proofs (to be answered on the pages they're written on). If needed, additional sheets of blank paper may be used to finish your solutions.

Calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed.

Questions 1 – 3 are each worth 4 marks.
Questions 4 – 7 are each worth 5 marks.
Questions 8 – 10 are each worth 6 marks.
Questions 11 – 12 are each worth 10 marks.
The total number of marks available is 70.

"It requires a very unusual mind to undertake the analysis of the obvious." - Alfred North Whitehead — Andrew Wiles

Question	Answer
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	

A. 4 mark questions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$(n-2022)f(n) - f(2022-n) = 2022$$

for all real numbers n. Find $f(2022)$.

Solution $f(2022) = 4086462$. First plug in $n = 2022$ to get:

$$\begin{aligned}(n-2022)f(n) - f(2022-n) &= 2022\\ 0 \cdot f(2022) - f(0) &= 2022\\ f(0) &= -2022\end{aligned}$$

Now plus in $n = 0$ to get

$$\begin{aligned}(n-2022)f(n) - f(2022-n) &= 2022\\ (-2022)f(0) - f(2022) &= 2022\\ (-2022 \cdot -2022) - f(2022) &= 2022\\ f(2022) &= (-2022 \cdot -2022) - 2022\\ &= (2022 \cdot 2021)\\ &= 4086462\end{aligned}$$

2. Let n be a positive integer and let m be the largest odd divisor of n. Find the sum of all n such that

$$n + 6 = m^2.$$

Solution: 9.

First notice that if is n is even the LHS is even and the right hand side is odd. We therefore can assume n is odd. Note that as m is a factor of n that $n+6$ is congruent to 6 mod m and m^2 is congruent to 0 mod m. Therefor m must be an odd factor of 6. Only 1 and 3 are options, and only $m = 3$ turns out to work. Giving $n = 9$ as the only solution

3. Using each of the digits 1, 2, 3, and 4 twice, write out an eight-digit number in which there is one digit between the 1's, two digits between the 2's, three digits between the 3's and four digits between the 4's.

Solution: One such example is 13142324.

B. 5 mark questions

4. Let $n = 20!$: How many positive integers are factors of n?

Solution: 38880.

We write 20! as a product of primes. $20! = 2^{17}3^{8}5^{4}7^{2}11^{1}13^{1}17^{1}19^{1}$. This lets us choose the number of 2s in 18 ways the number of 3s in 9 ways and so on. For a total of $18 \times 9 \times 5 \times 3 \times 2^4 = 38880$ ways.

5. The following fraction is a rational number. Write it in simplest form:

$$\frac{\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}}}{\sqrt{8+2\sqrt{7}} - \sqrt{8-2\sqrt{7}}}$$

Solution 1: Compute the four square roots individually.
$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$
$\sqrt{3-2\sqrt{2}} = \sqrt{2}-1$
$\sqrt{8+2\sqrt{7}} = 1+\sqrt{7}$
$\sqrt{8-2\sqrt{7}} = \sqrt{7}-1$
Substituting these in gives that our fraction is in fact 1

6. Evaluate the following intergral

$$\int_{-2}^{2} \frac{\cos x}{1+2^x}\,\mathrm{d}x$$

Solution: $\sin(2)$. We observe first that $\cos(x) = \cos(-x)$ and write:

$$\begin{aligned}
\int_{-2}^{2} \frac{\cos x}{1+2^x}\,\mathrm{d}x &= \int_{-2}^{0} \frac{\cos x}{1+2^x}\,\mathrm{d}x + \int_{0}^{2} \frac{\cos x}{1+2^x}\,\mathrm{d}x \\
&= \int_{0}^{2} \frac{\cos x}{1+2^{-x}}\,\mathrm{d}x + \int_{0}^{2} \frac{\cos x}{1+2^x}\,\mathrm{d}x \\
&= \int_{0}^{2} \frac{2^x\cos x}{2^x+1}\,\mathrm{d}x + \int_{0}^{2} \frac{\cos x}{1+2^x}\,\mathrm{d}x \\
&= \int_{0}^{2} \frac{(2^x+1)\cos x}{2^x+1}\,\mathrm{d}x \\
&= \int_{0}^{2} \cos x\,\mathrm{d}x
\end{aligned}$$

Which evaluates to sin(2)

7. Eight consecutive 3-digit positive integers have the property that each of them is divisible by their last digit. What is the smallest of these numbers?

Solution 841. The numbers either end in 1 through 8 or 2 through 9 (or else we'd have division byu zero). Call the numbers $x+1$, $x+2$,..., $x+8$ and observe that x must be divisible by all the last digits or equivalently by the lcm of all last digits. IF the last digits are 2 through 9 this is 2520 which isn't a factor of any three digit number (it's too large).

If the last digits are 1 through 8 then the lcm is 840. Which means the numbers must be 841 through 848.

C. 6 mark questions

8. Let a, b and c be real numbers such that

$$a^2 + 5b^2 + 2c^2 + 9 = 4ab + 2bc + 6c$$

Find the value of $a + b + c$.

Solution $a + b + c = 12$

$$\begin{aligned} a^2 + 5b^2 + 2c^2 + 9 &= 4ab + 2bc + 6c \\ a^2 - 4ab + 5b^2 - 2bc - 6c + 2c^2 + 9 &= 0 \\ (a - 2b)^2 + b^2 - 2bc + 2c^2 - 6c + 9 &= 0 \\ (a - 2b)^2 + (b - c)^2 + (c - 3)^2 &= 0 \end{aligned}$$

So $c = 3$, $b = c = 3$ and $a = 2b = 6$. So 12

9. Find the value of the following:

$$\frac{d^{101}}{dx^{101}} x^{100} \sin(100x)\Big|_{x=0}$$

Solution:$100 \cdot 101!$. Recall the Taylor series $\sin(x) = x - \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - ...$

Therefore $x^{100}\sin(100x) = 100x^{101} - \frac{100^3 x^{103}}{3!} + \frac{100^5 x^{105}}{5!} - \frac{100^7 x^{107}}{7!} + ...$

Differenting this 101 times leaves $100 \cdot 101!$

10. Steve the Unlucky has a 6 sided die with the magical property that whenever the die is rolled and gives a 6, the die vanishes and two more dice with the same property magically appear. Steve the Unlucky decides to play a simple game as follows: he will roll the die, and as long as new dice appear he will roll each of them exactly once. What is the probability that Steve the Unlucky keeps playing this game forever?

Solution: 0. Let p be the probability that the game eventually terminates! Then $p = \frac{5}{6} + \frac{1}{6}p^2$. This is because either we don't roll a six and the game immediately terminates or we do roll a six and the are left with two copies of the original game. This quadratic solves to $p = 1$ or $p = 5$. $p = 5$ is nonsensical as a probabilty so $p = 1$ and we're looking for $1 - p$.

D. Proof questions, 10 marks each

11. Let m be an irrational number and n be an integer greater than 1.
Prove that: $(m+\sqrt{m^2-1})^{\frac{1}{n}} - (m-\sqrt{m^2-1})^{\frac{1}{n}}$ is an irrational number.

Solution: This question had a typo and as stated the conclusion turns out to be false. The intended question was to show that $(m+\sqrt{m^2-1})^{\frac{1}{n}} + (m-\sqrt{m^2-1})^{\frac{1}{n}}$

We begin by proving a lemma that if $s-\frac{1}{s}$ is rational then $s^n+\frac{1}{s^n}$ is rational. This is shown by induction on n. More precisely we shall show that $s^{k+1}+\frac{1}{s^{k+1}}$ is rational using the assumed rationality of $s^k+\frac{1}{s^k}$ and $s^{k-1}+\frac{1}{s^{k-1}}$. We shall therefore require both $n=1$ and $n=2$ to be proven as the base case. The case of $n=1$ is trivial and we can handle the case of $n=2$ by observing that $s^2+\frac{1}{s^2}=(s+\frac{1}{s})^2-2$.

Proceeding to the inductive step we see that $s^{k+1}+\frac{1}{s^{k+1}}=(s^k+\frac{1}{s^k})\cdot(s+\frac{1}{s})-(s^{k-1}+\frac{1}{s^{k-1}})$. Which is enough to prove our lemma.

Moving on to the main part of the proof, we let $X=(m+\sqrt{m^2-1})^{\frac{1}{n}}+(m-\sqrt{m^2-1})^{\frac{1}{n}}$ and define $y=(m+\sqrt{m^2-1})^{\frac{1}{n}}$.

It is easy to check that $\frac{1}{y}=(m-\sqrt{m^2-1})^{\frac{1}{n}}$ so $X=y+\frac{1}{y}$. Which means that by our lemma if X is rational then so is $y^n+\frac{1}{y^n}=(m+\sqrt{m^2-1})+(m-\sqrt{m^2-1})=2m$, but $2m$ is given as irrational so it follows that $y+\frac{1}{y}$ is irrational.

12. The numbers $1, 2, 3, \ldots, 2n-1, 2n$ are arbitrarily divided into two groups with n numbers each. The numbers in the first group are written in ascending order, denoted by $a_1, a_2, \ldots, a_n$, and the numbers in the second group are written in descending order, denoted by $b_1, b_2, \ldots, b_n$. (So $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$.) Find, with proof the value of the following expression:

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$$

Solution n^2. Call the numbers 1, 2, 3,..., n 'small' and the numbers $n+1$, $n+2$, $n+3$,...,$2n$ 'big'. There must be exactly as many big numbers in A and small numbers in B and visa versa. Therefore every pair will be a big number minus a small number. This means that the total will always be $[(n+1) + (n+2) + ...(2n)] - [1 + 2 + ... + n] = n^2$