**2022 Wits Mathematics Competition**
**Final Round**
**Undergraduate**

### Instructions

This paper is 90 minutes long and consists of ten single answer questions (to be answered in the below table) and two proofs (to be answered on the pages they're written on). If needed, additional sheets of blank paper may be used to finish your solutions.

Calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed.

Questions 1 – 3 are each worth 4 marks.
Questions 4 – 7 are each worth 5 marks.
Questions 8 – 10 are each worth 6 marks.
Questions 11 – 12 are each worth 10 marks.
The total number of marks available is 70.

"It requires a very unusual mind to undertake the analysis of the obvious." - Alfred North Whitehead — Andrew Wiles

| Question | Answer |
| --- | --- |
| 1. | |
| 2. | |
| 3. | |
| 4. | |
| 5. | |
| 6. | |
| 7. | |
| 8. | |
| 9. | |
| 10. | |

# A. 4 mark questions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$(n - 2022)f(n) - f(2022 - n) = 2022$$

for all real numbers $n$. Find $f(2022)$.

2. Let $n$ be a positive integer and let $m$ be the largest odd divisor of $n$. Find the sum of all $n$ such that

$$n + 6 = m^2.$$

3. Using each of the digits 1, 2, 3, and 4 twice, write out an eight-digit number in which there is one digit between the 1's, two digits between the 2's, three digits between the 3's and four digits between the 4's.

## B. 5 mark questions

4. Let $n = 20!$: How many positive integers are factors of $n$?

5. The following fraction is a rational number. Write it in simplest form:

$$\frac{\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}}{\sqrt{8+2\sqrt{7}}-\sqrt{8-2\sqrt{7}}}$$

6. Evaluate the following intergral

$$\int_{-2}^{2} \frac{\cos x}{1+2^x}\ \mathrm{d}x$$

7. Eight consecutive 3-digit positive integers have the property that each of them is divisible by their last digit. What is the smallest of these numbers?

## C. 6 mark questions

8. Let $a$, $b$ and $c$ be real numbers such that

$$a^2 + 5b^2 + 2c^2 + 9 = 4ab + 2bc + 6c$$

Find the value of $a + b + c$.

9. Find the value of the following:

$$\left.\frac{d^{101}}{dx^{101}} x^{100} \sin(100x)\right|_{x=0}$$

10. Steve the Unlucky has a 6 sided die with the magical property that whenever the die is rolled and gives a 6, the die vanishes and two more dice with the same property magically appear. Steve the Unlucky decides to play a simple game as follows: he will roll the die, and as long as new dice appear he will roll each of them exactly once. What is the probability that Steve the Unlucky keeps playing this game forever?

# D. Proof questions, 10 marks each

11. Let $m$ be an irrational number and $n$ be an integer greater than 1.
Prove that: $(m+\sqrt{m^2-1})^{\frac{1}{n}}-(m-\sqrt{m^2-1})^{\frac{1}{n}}$ is an irrational number.

12. The numbers $1, 2, 3, \ldots, 2n-1, 2n$ are arbitrarily divided into two groups with $n$ numbers each. The numbers in the first group are written in ascending order, denoted by $a_1, a_2, \ldots, a_n$, and the numbers in the second group are written in descending order, denoted by $b_1, b_2, \ldots, b_n$. (So $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$.) Find, with proof the value of the following expression:

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$$