2022 Wits Mathematics Competition
Qualifying Round
Senior Secondary

Instructions

This exam consists of 20 multiple choice questions. There is one correct answer to each question. There is no penalty for incorrect answers. The mark allocation is as follows:

Questions 1-5 are each worth 3 points,
Questions 6-10 are each worth 4 points,
Questions 11-15 are each worth 5 points,
Questions 16-20 are each worth 6 points.
The total number of points available is 90.

The time limit on this exam is 75 minutes, calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed. If you are using the computer-friendly answer sheet you should fill it in in BLACK pen (other colours do not scan well). Time may be given for filling in name, school and other personal details.

It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profoundity, he is talking nonsense" — Alfred North Whitehead

A. 3 point questions

1. Compute $2022 - 1234$.

   A) 777 B) 788 C) 877 D) 886 E) 920

   Solution B: This can be done by subtraction or simply by looking at the last digit of the available options.

2. Katlego used 195 digits to number the pages in her diary. How many pages did she use?

   A) 87 B) 90 C) 99 D) 101 E) 195

   Solution D: 9 digits for pg $1-9$; 180 digits for pg $10-99$ and 3 digits for pg $100-101 = 101$ pages.

3. A water tank is $\frac{5}{6}$ full. When 30 litres is released the tank is $\frac{4}{5}$ full. Find the capacity of the tank in litres.

   A) 275 B) 300 C) 900 D) 1200 E) 1500

   Solution: C. Let the capacity be x then $\frac{5x}{6} - \frac{4x}{5} = 30$, which is easily solved to $x = 900$.

4. Bob adds up all the odd numbers from 1 to 1001 (including 1 and 1001). He then subtracts the sum of all the even numbers between 1 and 1001. What total does he end up with?

   A) 1002 B) 2004 C) 502 D) 998 E) 501

   Solution: E. The sum $1 - 2 + 3 - 4 + 5 - ... + 999 - 1000 + 1001$ may be split as follows $1 + (3 - 2) + (5 - 4) + .. + (1001 - 1000)$ which is 501.

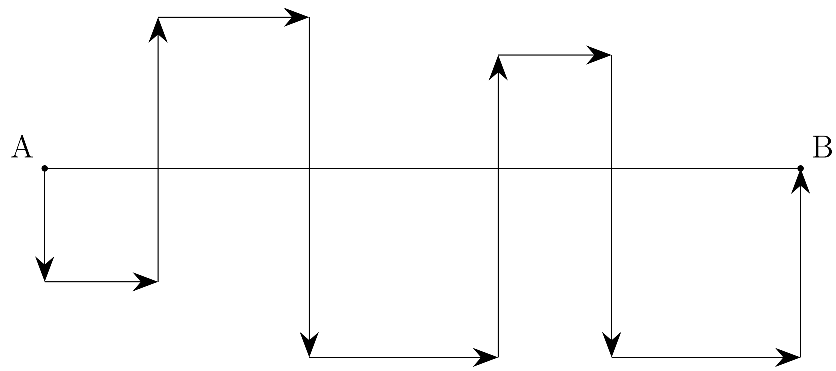
5. The product of two positive integers is equal to twice their sum. The same product is also equal to 6 times the difference between the two integers. What is the sum of the integers?

   A) 6 B) 9 C) 12 D) 15 E) 16

   Solution: B. Call the smaller number a and the larger one b. Then $2a + 2b = 6b - 6a$ which solves to $b = 2a$. Substituting into $ab = 2a + 2b$ gives $2a^2 = 6a$ and $a = 3$, $b = 6$. These sum to 9.

B. 4 point questions

6. A sequence of squares is shown, each square having a side on the line AB. A path is drawn, indicated by the arrows. The path is 96 metres long. Calculate the length AB.



A) 18 B) 24 C) 32 D) 36 E) 48

Solution: C. The line takes up a quarter of the perimeter of the squares, the path the other 3 quarters.

7. 30 students wrote a mathematics exam. The average score was 50. The average score of those who passed was 60, the average score of those who failed was 45. How many students passed?

A) 8 B) 10 C) 12 D) 16 E) 24

Solution: B. A total of 1500 points were scored. If x passed then $1500 = x(60) + (30 - x)(45)$ which solves to $x = 10$.

8. The multiplication $abc \times de = 7632$ uses each of the digits 1 to 9 exactly once. Calculate $a + b + c$.

A) 13 B) 14 C) 15 D) 16 E) 17

Solution: C. We begin by factorising 7632 into $2^4 \cdot 3^2 \cdot 53$. 53 must be a factor of abc or de. It can't be a factor of de as the 3 digit is already used and anything larger would require three digits. Hence abc must be a multiple of 53. We list the three digit multiples of 53: 106, 159, 212, 265, 318, 371, 424, 477, 530, 583, 636, 689, 742, 795, 848, 891, 944 and 997.

Removing those multiples with zeros, double digits or using a in use digit leaves: 159, 689 and 891. Of these only 159 fits the bill so the digit sum is 15

9. $ABCD$ is a rectangle. Point E lies on AB such that angle $DEC = 90^o$. $DC = \sqrt{10}$ cm and $DE = 3$ cm. Find the area of $ABCD$.

A) $2\sqrt{10}$ B) $\frac{3\sqrt{3}}{2}$ C) 3 D) 6 E) 12

Solution: C 3. Observe that ADE, DEC and BEC are similar triangles to get $BC = \frac{3}{\sqrt{10}}$.

10. A regular polygon with 6 sides has 9 diagonals. How many diagonals does a regular polygon with 10 sides have?

A) 25 B) 28 C) 30 D) 33 E) 35

Solution E: There are $\binom{10}{2} = 45$ ways to choose choose two vertices. However we must subtract the 10 non-diagonal lines which represent the sides to get 35. For those learners unfamiliar with binomial coefficents the 45 ways to choose two pairs can be subtracted by choosing 10 for the first point and 9 for the second and dividing by 2 because you have counted (a, b) and (b, a).

C. 5 point questions

11. Compute the product

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})...(1-\frac{1}{2021^2})(1-\frac{1}{2022^2}).$$

A) $\frac{1}{2}$ B) $\frac{2023}{4044}$ C) $\frac{1011}{2022}$ D) $\frac{2021^2}{4044^2}$ E) $\frac{5}{12}$

Solution B:

$$\begin{aligned}(1-\frac{1}{2^2})(1-\frac{1}{3^2})...(1-\frac{1}{2021^2})(1-\frac{1}{2022^2}) &= (\frac{2^2-1}{2^2})(\frac{3^2-1}{3^2})...(\frac{2021^2-1}{2021^2})(\frac{2022^2-1}{2022^2})\\ &= (\frac{1\cdot 3}{2\cdot 2})(\frac{2\cdot 4}{3\cdot 3})...(\frac{2020\cdot 2022}{2021\cdot 2021})(\frac{2021\cdot 2023}{2022\cdot 2022})\\ &= \frac{1\cdot 2023}{2\cdot 2022}\\ &= \frac{2023}{4044}\end{aligned}$$

12. How many zeroes will there be at the end of $1\times 2\times 3\times 4\times\cdots\times 39\times 40$ when multiplied out (when all the natural numbers from 1 to 40 are multiplied together)?

A) 6 B) 7 C) 8 D) 9 E) 11

Solution Option: D Each multiple of 5 adds an extra zero to the end and each multiple of 25 adds an additional zeroe. Hence $8+1=9$.

13. Find the sum of all values of x satisfying the following simultaneous equations:

$$\begin{aligned}x^2+3y &= 10,\\ 3+y &= \frac{10}{x}.\end{aligned}$$

A) -3 B) -2 C) 0 D) 4 E) 5

Solution: C. Note that we can write $y=3-\frac{10}{x}$. Substituting this into the other equation gives $x^3+30-9x=10x$ or $x^3-19x+30=0$. Which may be factorized as $(x-2)(x-3)(x+5)=0$ giving roots of $x=-2,-3$ and 5.

14. The sum of the digits of the integer equal to

$$777\ 777\ 777\ 777\ 777^2-222\ 222\ 222\ 222\ 223^2$$

is

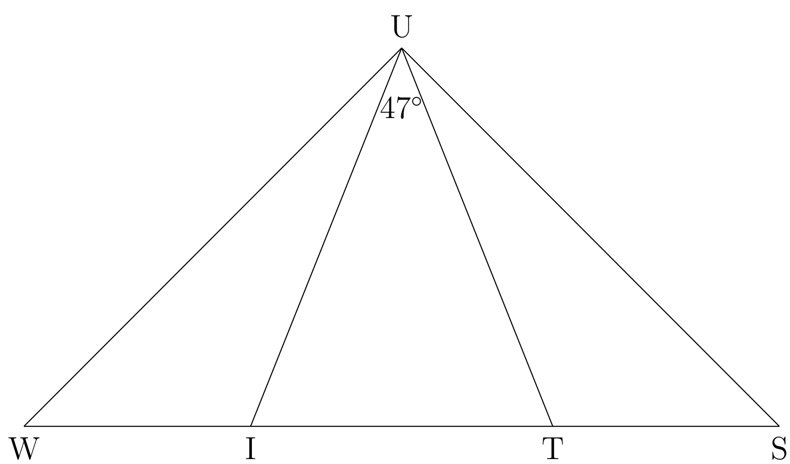
A) 148 B) 84 C) 74 D) 69 E) 79

Solution: C

$$\begin{aligned} 777\,777\,777\,777\,777^2 - 222\,222\,222\,222\,223^2 = \\ (777\,777\,777\,777\,777 + 222\,222\,222\,222\,223)(777\,777\,777\,777\,777 - 222\,222\,222\,222\,223) = \\ (1\,000\,000\,000\,000\,000)(555\,555\,555\,555\,554) = \\ 555\,555\,555\,555\,554\,000\,000\,000\,000\,000 \end{aligned}$$

These sum to 74.

15. In a triangle UWS the points I and T are placed on side WS so that $WU = WT$ and $SU = SI$. Determine $W\hat{U}S$ if $I\hat{U}T = 47^o$.



A) 82 B) 84 C) 86 D) 88 E) 90

Solution C. Set $W\hat{T}U = x$. Then angle chase to get, $W\hat{U}I = x - 47$,$T\hat{I}U = 133 - x$ and $T\hat{U}S = 86 - x$. Adding the three angles around U together gives a final answer of 86^o

D. 6 point questions

16. Three boys are aged 4, 6 and 7. Three girls are aged 5, 8 and 9. Two boys and two girls are selected randomly. Find the probability that the sum of the selected children's ages is even.

A) 0 B) $\frac{2}{9}$ C) $\frac{4}{9}$ D) $\frac{5}{9}$ E) 1

Solution: D $\frac{5}{9}$. Notice that the sum of all six childrens' ages is odd (as three of them are of odd ages. To get an even sum the ages of the two children not picked must add up to an odd number. Either an even-aged boy and an odd aged girl or an odd aged-girl and even aged boy. There are five ways to do this and nine ways to choose the two left out children.

17. $|x|$ is defined as x when $x > 0$ and $-x$ when $x < 0$. For example $|12| = 12$ and $|-4| = 4$. For real numbers x, y and z unequal to zero define the function

$$f(x, y, z) = 2 - \frac{|x|}{x} - 3\frac{|xy|}{xy} + 8\frac{|z|}{z}.$$

Find the difference between the maximum and minimum values that f can take on.

A) 18 B) 20 C) 22 D) 24 E) 26

Solution: D 24. The maximum is 14 and the minimum is -10.

18. How many positive integers below one thousand contain exactly three ones when written in binary? Equivalently how many positive integers below one thousand can be written as a sum of three different powers of two.

A) 90 B) 120 C) 150 D) 180 E) 210

Solution: B 120. The question can be reduced to choosing three positions for the ones in a ten digit binary expansion with leading zeros allowed. There are $\binom{10}{3} = 120$ ways to do this.

19. Find the sum of the two smallest positive integers with exactly 10 distinct factors.

A) 48 B) 128 C) 210 D) 512 E) 560

Answer: B 128

Our options for numbers with ten factors are those numbers of the form p^9 or pq^4 where p and q are distinct primes for which we wish to minimise. Try 2^9 and 3×2^4 and we get 48 is the smallest and $80 = 5 \times 2^4$.

20. Let $[n] = \{1, 2, \ldots, n\}$. For T a non-empty subset of $[n]$ let M_T be the reciprocal of the product of elements of T. Define S_n as the sum of M_T across all such non-empty subsets of $[n]$. Find S_{2022}.

A) 1621 B) 1811 C) 2022 D) 4044 E) 6000

Solution: D 2022. More generally it can be shown by induction that $S_n = n$. Once one observes the recurence $S_{n+1} = S_n + \frac{1}{n+1} S_n + \frac{1}{n+1}$ this becomes straight forward.