2022 Wits Mathematics Competition
Final Round
Senior Secondary

Instructions

This paper is 90 minutes long and consists of ten single answer questions (to be answered in the below table) and two proofs (to be answered on the pages they're written on).

If needed, additional sheets of blank paper may be used to finish your solutions.

Calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed.

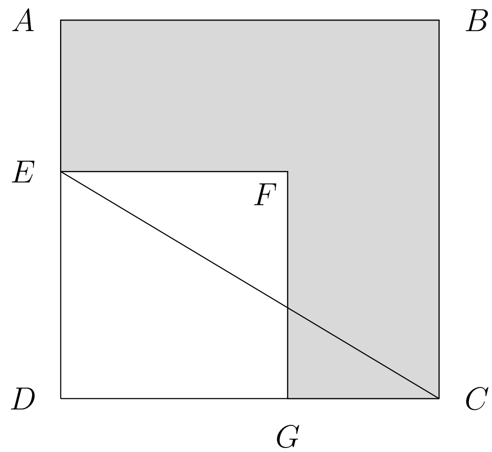
Questions 1 – 3 are each worth 4 marks.
Questions 4 – 7 are each worth 5 marks.
Questions 8 – 10 are each worth 6 marks.
Questions 11 – 12 are each worth 10 marks.
The total number of marks available is 70.

"It requires a very unusual mind to undertake the analysis of the obvious." - Alfred North Whitehead

Question	Answer
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	

A. 4 point questions

1. $ABCD$ and $DEFG$ are squares, $CE = 17$cm and the shaded region $ABCGFE$ is 161cm^2. What is the perimeter of the shaded region?



Solution 24.

Let $ABCD$ have side length s and $DEFG$ have side length t. The perimeter of the shaded region is $3s$ as $EF + GC = s = AE + FG$. Now by pythagorus $s^2 + t^2 = 289$ and we are given $s^2 - t^2 = 161$. Adding these together gives $2s^2 = 128$ which solves to $s = 8$. Therefore the perimeter is 24.

2. What is the maximum value of the expression $(1 + x)(1 - x) + (3 - y)(2 + y)$ where x and y are real numbers?

Solution $\frac{29}{4}$:

The trick here is to notice that we can optimize in x and y seperately here as the terms don't interact.

$$\begin{aligned}(1 + x)(1 - x) + (3 - y)(2 + y) &= (1 - x^2) + (6 + y - y^2) \\ &= (1 - x^2) + (\frac{25}{4} - (\frac{1}{2} - y)^2)\end{aligned}$$

Which is maximized by setting the squares equal to zero. This gives a final answer of $\frac{29}{4}$.

3. What is the median value for x such that $\lfloor 3x + 4 \rfloor = 1$? Here $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

Solution $-\frac{5}{6}$. $\lfloor 3x + 4 \rfloor = 1$ when $x \in [-1, -\frac{2}{3})$.

B. 5 mark questions

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$(n - 2022)f(n) - f(2022 - n) = 2022$$

for all real numbers n. Find $f(2022)$.

Solution $f(2022) = 4086462$. First plug in $n = 2022$ to get:

$$\begin{aligned} (n - 2022)f(n) - f(2022 - n) &= 2022 \\ 0 \cdot f(2022) - f(0) &= 2022 \\ f(0) &= -2022 \end{aligned}$$

Now plus in $n = 0$ to get

$$\begin{aligned} (n - 2022)f(n) - f(2022 - n) &= 2022 \\ (-2022)f(0) - f(2022) &= 2022 \\ (-2022 \cdot -2022) - f(2022) &= 2022 \\ f(2022) &= (-2022 \cdot -2022) - 2022 \\ &= (2022 \cdot 2021) \\ &= 4086462 \end{aligned}$$

5. Adele and Bongi play a game. The game ends when either Adele or Bongi win two consecutive rounds. The probability that Adele wins a round is 0.3, while the probability that Bongi wins a round is 0.6. The probability that a round is drawn is 0.1. What is the probability that neither is the winner after at most three games?

Solution: 0.343

There are a few ways to not have a winner after three rounds. Either have a draw in the first round and don't have either play win the next two $0.1 \times (1 - 0.3^2 - 0.6^2) = 0.055$, have have a draw first but have a draw second $0.9 \times 0.1 = 0.09$ or $ABA = 0.054$, $BAB = 0.108$, $ABD = 0.018$ or $BAD = 0.018$. These sum to 0.343

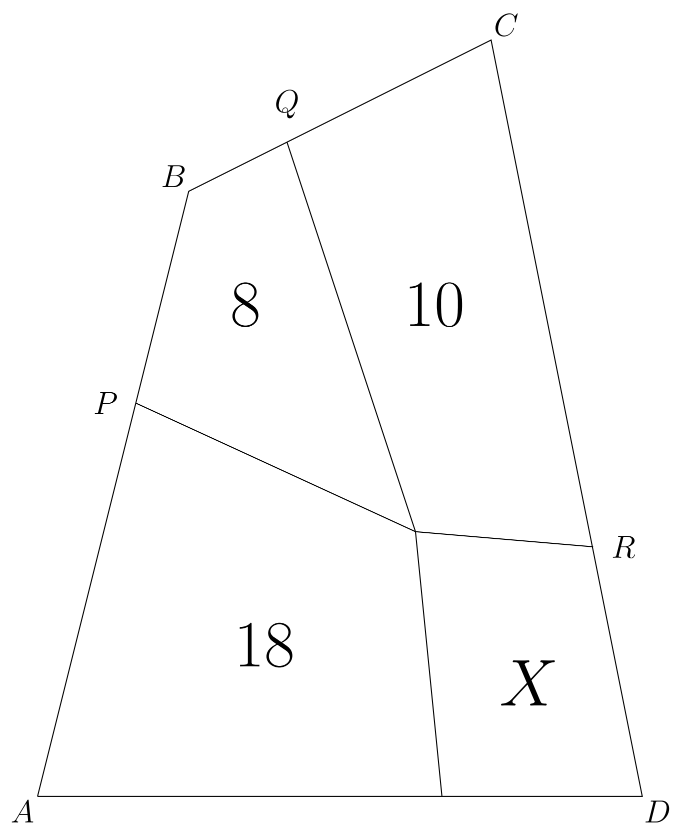
6. The following fraction is a rational number. Write it in its simplest form:

$$\frac{\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}}{\sqrt{8+2\sqrt{7}}-\sqrt{8-2\sqrt{7}}}$$

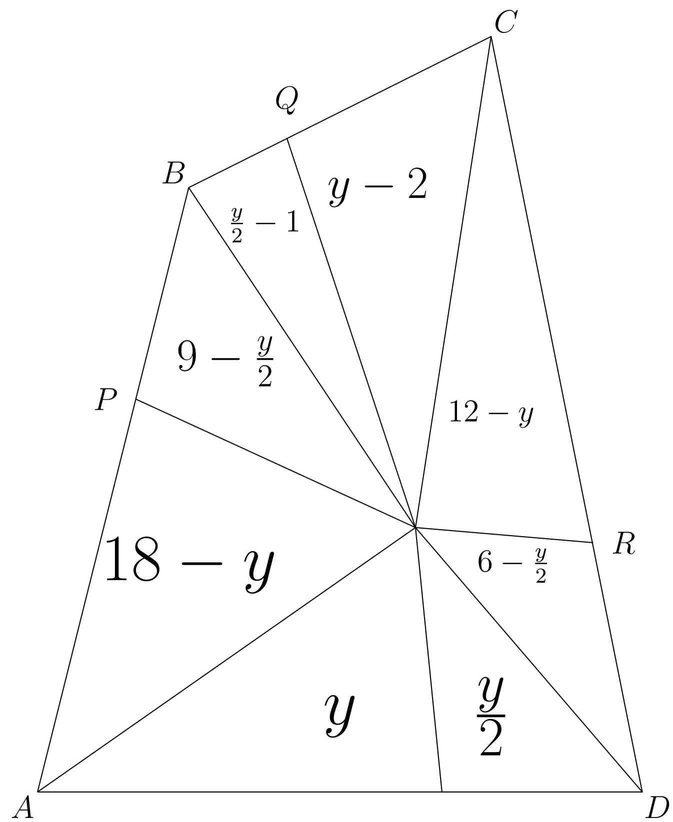
Solution 1:

Compute the four square roots individually.
$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$
$\sqrt{3-2\sqrt{2}} = \sqrt{2}-1$
$\sqrt{8+2\sqrt{7}} = 1+\sqrt{7}$
$\sqrt{7-2\sqrt{7}} = \sqrt{7}-1$
Substituting these in gives that our fraction is in fact 1

7. $ABCD$ is a quadrilateral and this is divided into four smaller quadrilaterals with M as a common vertex, and the areas are as indicated in the diagram. $AP : PB = 2 : 1$, $CQ : QB = 2 : 1$, $CR : RD = 2 : 1$, and $AS : SD = 2 : 1$. Calculate the area X.



Solution: 6. We draw in lines from the four corners to the meeting point M and let the area of triangle AMS be y. It is then easy to calculat the areas of the other drawn in triangles in terms of y as seen below.



C. 6 mark questions

8. What is the sum of the integers k for which the expression below is also an integer?

$$\frac{(k^2+2k-6)^2}{k+1}$$

Solution: -6

$$\begin{aligned}\frac{(k^2+2k-6)^2}{k+1} &= \frac{(k^2+2k+1-5)^2}{k+1}\\ &= \frac{((k+1)^2-5)^2}{k+1}\\ &= \frac{(k+1)^4-10(k+1)^2+25}{k+1}\\ &= (k+1)^3-10(k+1)+\frac{25}{k+1}\end{aligned}$$

Which is an integer if and only if $k+1$ dives 25. That is when $k+1 \in \{1, 5, 25, -1, -5, -25\}$ or equivalently $k \in \{0, 4, 24, -2, -6, -26\}$.

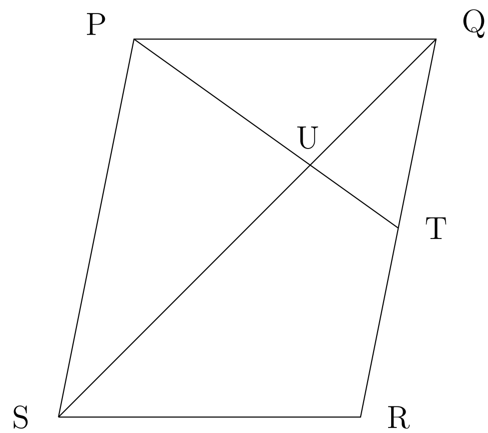
Which sums to -6

9. Eight consecutive 3-digit positive integers have the property that each of them is divisible by their last digit. What is the smallest of these numbers?

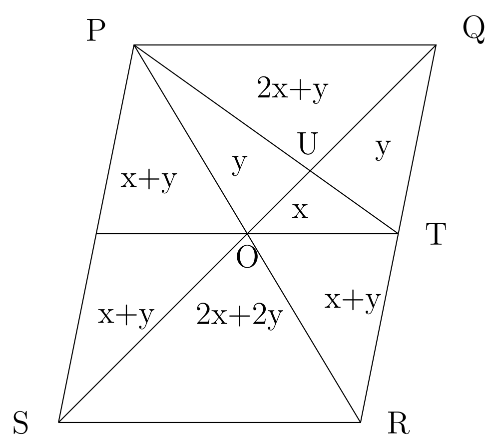
Solution 841. The numbers either end in 1 through 8 or 2 through 9 (or else we'd have division byu zero). Call the numbers $x+1$, $x+2$,..., $x+8$ and observe that x must be divisible by all the last digits or equivalently by the lcm of all last digits. IF the last digits are 2 through 9 this is 2520 which isn't a factor of any three digit number (it's too large).

If the last digits are 1 through 8 then the lcm is 840. Which means the numbers must be 841 through 848.

10. $PQRS$ is a parallelogram and T is the mid-point of QR. PT and SQ intersect at U. What is the ratio of the area of $SRTU$ to the area of $PQRS$?



Solution: $\frac{5}{12}$. Draw in the line segment PR and call the oint where PR and SQ intersect O and extend the segment TO to PS. Then call the area of triangle OUT x and the area of triangle QUT y. From here it is possible to find the areas of other triangles by comparing bases and heights. They are shown in the diagram below.



Finally observe from the diagram that $\frac{y}{2x+y} = \frac{x}{y}$ which implies $y = 2x$. Out final ratio then is $\frac{4x+3y}{8x+8y} = \frac{10x}{24x} = \frac{5}{12}$

D. Proof questions, 10 marks each

11. The numbers 1, 2, 3, $\ldots$, $2n-1$, and $2n$ are arbitrarily divided into two groups with n numbers each. The numbers in the first group are written in ascending order, denoted by $a_1, a_2, \ldots, a_n$, and the numbers in the second group are written in descending order, denoted by $b_1, b_2, \ldots, b_n$. (So $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$.)

    Find, with proof the value of the following expression:

    $$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$$

    Solution n^2. Call the numbers 1, 2, 3,..., n 'small' and the numbers $n+1$, $n+2$, $n+3$,...,$2n$ 'big'. There must be exactly as many big numbers in A and small numbers in B and visa versa. Therefore every pair will be a big number minus a small number. This means that the total will always be $[(n+1)+(n+2)+...(2n)] - [1+2+...+n] = n^2$

12. Let m be an irrational number greater than 1 and n be an integer greater than 1. Prove that $(m+\sqrt{m^2-1})^{\frac{1}{n}} - (m-\sqrt{m^2-1})^{\frac{1}{n}}$ is an irrational number.

Solution: This question had a typo and as stated the conclusion turns out to be false. The intended question was to show that $(m+\sqrt{m^2-1})^{\frac{1}{n}} + (m-\sqrt{m^2-1})^{\frac{1}{n}}$

We begin by proving a lemma that if $s - \frac{1}{s}$ is rational then $s^n + \frac{1}{s^n}$ is rational. This is shown by induction on n. More precisely we shall show that $s^{k+1} + \frac{1}{s^{k+1}}$ is rational using the assumed rationality of $s^k + \frac{1}{s^k}$ and $s^{k-1} + \frac{1}{s^{k-1}}$. We shall therefore require both $n = 1$ and $n = 2$ to be proven as the base case. The case of $n = 1$ is trivial and we can handle the case of $n = 2$ by observing that $s^2 + \frac{1}{s^2} = (s + \frac{1}{s})^2 - 2$.

Proceeding to the inductive step we see that $s^{k+1} + \frac{1}{s^{k+1}} = (s^k + \frac{1}{s^k}) \cdot (s + \frac{1}{s}) - (s^{k-1} + \frac{1}{s^{k-1}})$. Which is enough to prove our lemma.

Moving on to the main part of the proof, we let $X = (m+\sqrt{m^2-1})^{\frac{1}{n}} + (m-\sqrt{m^2-1})^{\frac{1}{n}}$ and define $y = (m+\sqrt{m^2-1})^{\frac{1}{n}}$.

It is easy to check that $\frac{1}{y} = (m-\sqrt{m^2-1})^{\frac{1}{n}}$ so $X = y + \frac{1}{y}$. Which means that by our lemma if X is rational then so is $y^n + \frac{1}{y^n} = (m+\sqrt{m^2-1}) + (m-\sqrt{m^2-1}) = 2m$, but $2m$ is given as irrational so it follows that $y + \frac{1}{y}$ is irrational.