2022 Wits Mathematics Competition
Final Round
Senior Secondary

Instructions

This paper is 90 minutes long and consists of ten single answer questions (to be answered in the below table) and two proofs (to be answered on the pages they're written on).

If needed, additional sheets of blank paper may be used to finish your solutions.

Calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed.

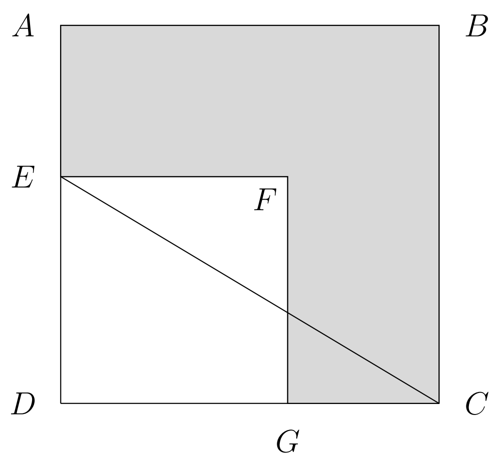
Questions 1 – 3 are each worth 4 marks.
Questions 4 – 7 are each worth 5 marks.
Questions 8 – 10 are each worth 6 marks.
Questions 11 – 12 are each worth 10 marks.
The total number of marks available is 70.

"It requires a very unusual mind to undertake the analysis of the obvious." - Alfred North Whitehead

Question	Answer
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	

A. 4 point questions

1. $ABCD$ and $DEFG$ are squares, $CE = 17$cm and the shaded region $ABCGFE$ is 161cm^2. What is the perimeter of the shaded region?



2. What is the maximum value of the expression $(1 + x)(1 - x) + (3 - y)(2 + y)$ where x and y are real numbers?

3. What is the median value for x such that $\lfloor 3x + 4 \rfloor = 1$? Here $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

B. 5 mark questions

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$(n - 2022)f(n) - f(2022 - n) = 2022$$

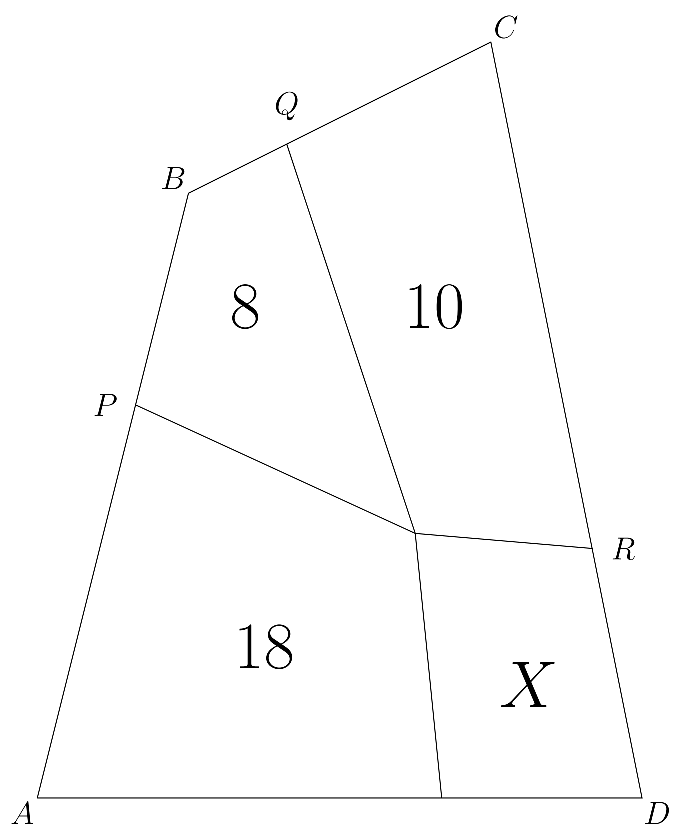
for all real numbers n. Find $f(2022)$.

5. Adele and Bongi play a game. The game ends when either Adele or Bongi win two consecutive rounds. The probability that Adele wins a round is 0.3, while the probability that Bongi wins a round is 0.6. The probability that a round is drawn is 0.1. What is the probability that neither is the winner after at most three games?

6. The following fraction is a rational number. Write it in its simplest form:

$$\frac{\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}}}{\sqrt{8+2\sqrt{7}} - \sqrt{8-2\sqrt{7}}}$$

7. $ABCD$ is a quadrilateral and this is divided into four smaller quadrilaterals with M as a common vertex, and the areas are as indicated in the diagram. $AP : PB = 2 : 1$, $CQ : QB = 2 : 1$, $CR : RD = 2 : 1$, and $AS : SD = 2 : 1$. Calculate the area X.



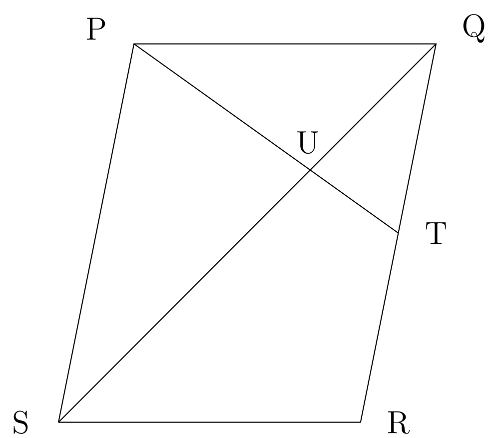
C. 6 mark questions

8. What is the sum of the integers k for which the expression below is also an integer?

$$\frac{(k^2+2k-6)^2}{k+1}$$

9. Eight consecutive 3-digit positive integers have the property that each of them is divisible by their last digit. What is the smallest of these numbers?

10. $PQRS$ is a parallelogram and T is the mid-point of QR. PT and SQ intersect at U. What is the ratio of the area of $SRTU$ to the area of $PQRS$?



D. Proof questions, 10 marks each

11. The numbers 1, 2, 3, ..., $2n-1$, and $2n$ are arbitrarily divided into two groups with n numbers each. The numbers in the first group are written in ascending order, denoted by $a_1, a_2, \ldots, a_n$, and the numbers in the second group are written in descending order, denoted by $b_1, b_2, \ldots, b_n$. (So $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$.)

    Find, with proof the value of the following expression:

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$$

12. Let m be an irrational number greater than 1 and n be an integer greater than 1. Prove that $(m + \sqrt{m^2 - 1})^{\frac{1}{n}} - (m - \sqrt{m^2 - 1})^{\frac{1}{n}}$ is an irrational number.