2022 Wits Mathematics Competition
Final Round
Junior Secondary

Instructions

This paper is 90 minutes long and consists of ten single answer questions (to be answered in the below table) and two proofs (to be answered on the pages they're written on).

If needed, additional sheets of blank paper may be used to finish your solutions.

Calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed.

Questions 1 – 3 are each worth 4 marks.
Questions 4 – 7 are each worth 5 marks.
Questions 8 – 10 are each worth 6 marks.
Questions 11 – 12 are each worth 10 marks.
The total number of marks available is 70.

"It requires a very unusual mind to undertake the analysis of the obvious." - Alfred North Whitehead

Question	Answer
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	

A. 4 point questions

1. Which number must replace the question mark if the total of the numbers in each row is the same?

1	2	3	4	5	6	7	8	9	10	199
11	12	13	14	15	16	17	18	19	20	?

   Solution 99.

   Each of the first ten numbers in the top row is 10 less than it's counterpart. To balance this the ? must be 100 less than it's counterpart.

2. Today is a Monday. Thabo starts to read a book with 290 pages today. On Mondays he reads 25 pages and on every other day he reads 4 pages. On which day of the week does he finish reading the book?

   Solution: A Saturday.

   Observe that Thabo reads a total of 49 pages a week. After six weeks (ending in a Sunday) he'd have read 296 pages if the book was long enough. The Saturday before he'd be on 292 pages and the Friday before on 288. Therefore he finishes on the sixth Saturday.

3. In a football match between Kaizer Chiefs and Orlando Pirates, the final score is 3 : 2. How many possible scorelines are there at half-time?

   Solution: 12.

   Chiefs could have a score of any of the four numbers 0, 1, 2 or 3 and Pirates could have any of the scores 0, 1 or 2 with no score by one eliminating scores by the other.

B. 5 mark questions

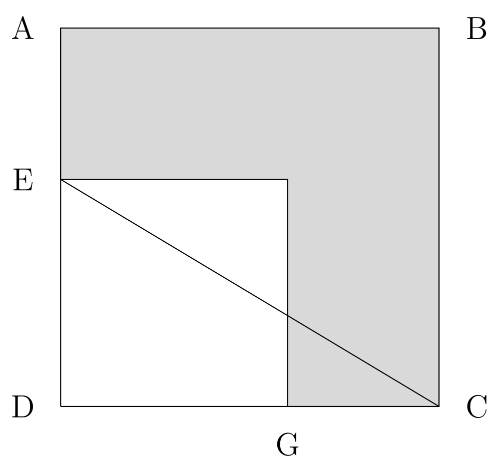
4. How many whole numbers between 2000 and 3000 have digits that sum to 6?

Solution: 15 The easiest way to solve this is to list them systematically. While it can also be solved by the so called 'stars and bars' method that requires knowing more mathematics than can reasonably be expected from junior secondary school students. However it's very much recommended that interested participants read up on it.

It's worth mentioning that the 'systematic' way below is to always place the largest digit you can (haven't before first). This gives a list in reverse order.

The possible answers are 2400, 2310, 2301, 2220, 2211, 2202, 2130, 2121, 2112, 2103, 2040, 2031, 2022, 2013 and 2004

5. ABCD and DEFG are squares. $CE = 17cm$ and the area of the shaded region ABCGFE is $161cm^2$. What is the perimeter of the shaded region in cm?



Solution 24. Let $ABCD$ have side length s and $DEFG$ have side length t. The perimeter of the shaded region is $3s$ as $EF+GC = s = AE+FG$. Now by pythagorus $s^2+t^2 = 289$ and we are given $s^2 - t^2 = 161$. Adding these together gives $2s^2 = 128$ which solves to $s = 8$. Therefore the perimeter is 24.

6. How many positive integers are factors of 20! (Note: ! denotes multiplying by one less number each time until you get to 1. For example, 4! is $4 \times 3 \times 2 \times 1$)?

   Solution: 38880.

   We write 20! as a product of primes. $20! = 2^{17}3^8 5^4 7^2 11^1 13^1 17^1 19^1$. This lets us choose the number of 2s in 18 ways the number of 3s in 9 ways and so on. For a total of $18 \times 9 \times 5 \times 3 \times 2^4 = 38880$ ways

7. Adele and Bongi play a game. The game ends when either Adele or Bongi win two consecutive rounds. The probability that Adele wins a round is 0.3, while the probability that Bongi wins a round is 0.6. The probability that a round is drawn is 0.1. What is the probability that neither is the winner after at most three games?

   Solution: 0.343

   There are a few ways to not have a winner after three rounds. Either have a draw in the first round and don't have either play win the next two $0.1 \times (1 - 0.3^2 - 0.6^2) = 0.055$, have have a draw first but have a draw second $0.9 \times 0.1 = 0.09$ or $ABA = 0.054$, $BAB = 0.108$, $ABD = 0.018$ or $BAD = 0.018$. These sum to 0.343

C. 6 mark questions

8. Ten people form a circle. Each picks a random number and tells it to the two neighbours adjacent to them in the circle. Then each person computes and announces the average of the numbers of their two neighbours. The figure shows the average announced by each person (not the original number the person picked!)

"1"
"10" "2"
"9" "3"
"8" "4"
"7" "5"
"6"

What was the number picked by the person who announced the average of 6?

Solution: 1. Let the number person six chooses be x. Then the number person four choose was $10 - x$, the number person two choose was $x - 4$ the number person ten choose was $6 - x$ and the number person eight choose was $x + 12$. Person six and eight's numbers add up to 14 so $2x + 12 = 14$ yields $x = 1$.

9. The stick of length 2.5m is split into two pieces. The break point is equally likely to be anywhere along the stick's length. After this the lengths of the two sticks are measured and the lengths (in meters) are rounded to the nearest integer. What is the probability that the two rounded integers add up to 3?

Solution: $\frac{2}{5}$.

This happens if and only if the shorter half is between 0.5m and 1m.

10. Let P_1 be a regular polygon with r sides and P_2 be a regular polygon with s sides, with $r \geq s \geq 3$. Each interior angle of P_1 is $\frac{59}{58}$ as large as each interior angle of P_2. What is the largest possible value of s?

Solution: 117. Recall that that internal angle of a regular polygon with r sides is $\frac{180(r-2)}{r}$.

This leads to the equation

$$\frac{59}{58} \cdot \frac{s-2}{s} = \frac{r-2}{r} < 1$$

So clearly $\frac{s-2}{s} < \frac{58}{59}$ and $s < 118$. We substitute in $s = 117$ and see that this gives an integer r as follows:

$$\begin{aligned}
\frac{59}{58} \cdot \frac{115}{117} &= \frac{r-2}{r} \\
1 - \frac{59}{58} \cdot \frac{115}{117} &= \frac{2}{r} \\
\frac{117 \cdot 58 - 59 \cdot 115}{58 \cdot 117} &= \frac{2}{r} \\
r &= \frac{2 \cdot 58 \cdot 117}{117 \cdot 58 - 59 \cdot 115} \\
r &= \frac{2 \cdot 58 \cdot 117}{115 \cdot 58 + 2 \cdot 58 - 58 \cdot 115 - 115} \\
r &= \frac{2 \cdot 58 \cdot 117}{2 \cdot 58 - 115} \\
r &= \frac{2 \cdot 58 \cdot 117}{1} \\
r &= 2 \cdot 58 \cdot 117
\end{aligned}$$

D. Proof questions, 10 marks each

11. Each of the letters in the equation $FORTY + TEN + TEN = SIXTY$ represents a different digit from 0 through 9. The digits are concatenated so for example if $A = 1$, $N = 3$ and $D = 8$, then $AND = 138$. Further in our example letters always mean the same thing. So the Y in $FORTY$ is the same digit as the Y in $SIXTY$. Find which letter corresponds to which number so that the equation is true. Show that your solution is the only possible solution. Show all work

Solution : $29786 + 850 + 850 = 31486$. We begin by observing that $E = 5$ and $N = 0$. This is the only way to have $FORTY$ and $SIXTY$ end in the same two digits. This means that out hundreds column is $R + 2T + 1$ which is at most 27. As we've used up 0 we need to have $O = 9$ and $I = 1$. Further we now need $R + 2T + 1$ to be at least 22 (If less than 20 we couldn't have enough separation between O and I and we can't have $X \in \{0, 1\}$ as both are taken. The only ways to do this are $T = 8$, $R = 7$ and $T = 8$, $R = 6$. Either way we get $T = 8$.

So our current set up is $F9R8Y + 850 + 850 = S1X8Y$ with the digits 2, 3, 4, 6 and 7 unused. Also we have that either $R = 6$,$X = 3$ or $R = 7$, $X = 4$. If we the former the unclaimed digits would be 2, 4 and 7 and it would be impossible to have $S = F + 1$. So we get $R = 7$ and $X = 4$. This gives $F = 2$, $S = 3$ and $Y = 6$.

12. What is the sum of the integers k for which the expression below is also an integer?

$$\frac{(k^2+2k-6)^2}{k+1}$$

Solution: -6

$$\begin{aligned}\frac{(k^2+2k-6)^2}{k+1} &= \frac{(k^2+2k+1-5)^2}{k+1} \\ &= \frac{((k+1)^2-5)^2}{k+1} \\ &= \frac{(k+1)^4-10(k+1)^2+25}{k+1} \\ &= (k+1)^3-10(k+1)+\frac{25}{k+1}\end{aligned}$$

Which is an integer if and only if $k+1$ dives 25. That is when $k+1 \in \{1, 5, 25, -1, -5, -25\}$ or equivalently $k \in \{0, 4, 24, -2, -6, -26\}$.

Which sums to -6