2021 Wits Mathematics Competition
Final Round
Undergraduate

Instructions

This paper is 90 minutes long and consists of ten single answer questions (to be answered in the below table) and two proofs (to be answered on the pages they're written on). If needed, additional sheets of blank paper may be used to finish your solutions.

Calculators may NOT be used. A ruler and compass may be used but all other geometric aids are NOT allowed. A translation aid (such as a dictionary) from English to another language is allowed.

Questions 1 – 3 are each worth 2 marks.
Questions 4 – 7 are each worth 3 marks.
Questions 8 – 10 are each worth 4 marks.
Questions 11 – 12 are each worth 10 marks.
The total number of marks available is 50.

"Always try the problems that matter most to you". — Andrew Wiles

Question	Answer
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	

A. 2 mark questions

1. What is the unit digit of 7^{4000}?

   Solution: First, we calculate the unit digit of the first few powers of seven: $7^1 \equiv 7$, $7^2 \equiv 9$,$7^3 \equiv 3$, $7^4 \equiv 1$, $7^5 \equiv 7 \ldots$. Here we see that the unit digit of each power of seven creates the pattern $7, 9, 3, 1$ and then keeps on repeating itself. Thus, 7^{4000} will have unit digit 1.

2. If i is the imaginary complex number, i.e., $i = \sqrt{-1}$, what will $(1 + i)^{16}$ equate to?

   Solution: We will first calculate $(1 + i)^2$ to simplify calculations. Thus, $(1 + i)^2 = (1+i)(1+i) = 1+2i+i^2 = 1+2i-1 = 2i$. Therefore, $(1+i)^{16} = ((1+i)^2)^8 = (2i)^8 = 2^8 i^8$. Since $i^2 = -1$, $i^8 = (i^2)^4 = (-1)^4 = 1$ and so we have $(1 + i)^{16} = 2^8 = 256$.

3. How many times must all three sides of an equilateral triangle be halved so that the area of the equilateral triangle obtained in the end is a 256th of the original equilateral triangle's area?

   Solution: Note that if the sides are halved of an equilateral triangle, then the area of the new equilateral triangle will have a quarter of the original triangle's area. (This can be seen by comparing the area of both triangles with each other.) Since, $1/256 = 1/4^4$, we only need to halve the sides of the original triangle 4 times to obtain the required result.

B. 3 mark questions

4. Given a 5 by 5 grid, what is the minimum number of blocks you need to fill in with a cross so that the following property holds: Filling in any empty block with a cross results in 3 crosses lying on a horizontal, vertical or diagonal straight line.

Figure 1: 5 by 5 grid

Solution: 6, for example:

		×		×
×				×
×		×		

Figure 2: Solution to 5 by 5 grid question.

5. The women's 200m breaststroke at the Olympics has 30 entrants. In order to determine the medal winners, that is, the 3 fastest swimmers, a number of heats are required. Each heat can have at most 6 swimmers.

   Unfortunately, the timer is broken so the times of the swimmers in each heat is unavailable. What is the minimum number of heats required to determine the medal winners?

   Solution: 7

   Explanation: Five heats needed to give everyone a chance to race, then one race amongst these group winners to rank each group and identify the overall winner. For second place, it will either be the second swimmer of the 6th race (in which case third could either be third in the 6th race or second in the overall winner's heat or second in the second fastest group) or position 2 of the initial heat which contained the overall winner (in which case third overall would be either second in the 6th race or third in the heat with the overall winner). So for the last race, put these five swimmers in the pool to finalise second and third place.

6. Given a regular hexagon circumscribed by a circle which is circumscribed by another regular hexagon, what is the area of the outer hexagon, if the area of the inner hexagon is 1?

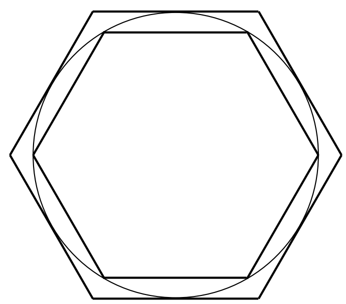

Figure 3: Hexagons and circle

Solution: 4/3

Explanation: By rotating the inner hexagon by 30 degrees, and splitting it into six equilateral triangles, each of which is split into three isosceles triangles, it can be seen that the ratio is 3:4 since the extra triangle which is in the outer hexagon but not the inner one is congruent to the smaller ones.

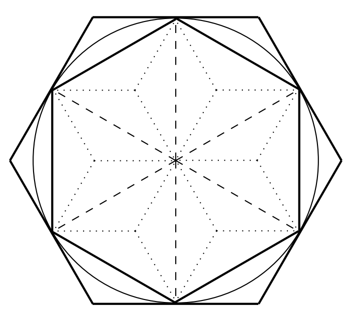

Figure 4: Hexagons and circle solution

(from Ivan Moskovich's Little Book of Big Brain Games)

7. How many pairs of integers a, b, with $a \geq 0$ and $b \geq 0$, will satisfy the equation

$$a^2 + 1 = 2^b.$$

Solution: 2

The two solutions are $a = 0$, $b = 0$ and $a = 1$, $b = 1$. For $a \geq 2$ there are no solutions, which we can show as follows:

If a is even then a^2 is even so $a^2 + 1$ is odd and cannot equal 2^b.

If a is odd, we consider two cases: $a = 4k + 1$ for some $k \in \mathbb{Z}$, and $a = 4k + 3$ for some $k \in \mathbb{Z}$.

In the first case, $a^2 + 1 = (4k + 1)^2 + 1 = 16k^2 + 8k + 2 = 2(8k^2 + 4k + 1)$. Thus, $a^2 + 1$ has an odd factor, namely $8k^2 + 4k + 1$. If $8k^2 + 4k + 1 \neq 1$ then it's clear that $a^2 + 1 \neq 2^b$. If $8k^2 + 4k + 1 = 1$ then, solving, we get $k = 0$, in which case $a = 1$ which is already counted.

The case in which $a = 4k + 3$ is similar. Thus, there are no solutions with $a \geq 2$.

C. 4 mark questions

8. Find $\dfrac{d^{10}}{dx^{10}}\sin(x^2)$ evaluated at $x = 0$.

Solution: Trying to differentiate the function directly 10 times is an extremely tedious process that takes a lot of time and leaves one prone to making errors. It is just not a practical approach to the problem in an olympiad setting.

Instead, one should expand the function in a MacLaurin series around 0:

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{71}...$$

Differentiating 10 times and evaluating at $x = 0$ gives:

$$\frac{d^{10}f}{dx^{10}}|_{x=0} = \frac{10!}{5!} = 30240$$

9. Evaluate the following in terms of b:

$$\frac{d}{db}\int_1^b \frac{\cos(bx)}{x}dx$$

Solution: We treat the integral as involving two unknowns, a and b:

$$I = \frac{d}{db}\int_1^a \frac{\cos(bx)}{x}dx$$

Using the multivariable chain rule:

$$\begin{aligned}
I &= \frac{\partial}{\partial a}\int_1^a \frac{\cos(bx)}{x}dx\frac{da}{db} + \frac{\partial}{\partial b}\int_1^a \frac{\cos(bx)}{x}dx \\
&= \frac{\cos ba}{a}\frac{da}{db} + \int_1^a \frac{\partial}{\partial b}\frac{\cos(bx)}{x}dx \\
&= \frac{\cos ba}{a}\frac{da}{db} - \int_1^a \sin(bx)dx \\
&= \frac{\cos ba}{a}\frac{da}{db} + \frac{1}{b}(\cos(ba) - \cos b)
\end{aligned}$$

Substituting $b = a$:

$$I = \frac{2\cos(b^2) - \cos b}{b}$$

10. Given the set $S = \{1, 2, 3, \ldots, 2021\}$ we seek a subset of S such that no two elements in the subset differ by 4 and no two elements differ by 7. What is the cardinality of the largest subset satisfying these conditions?

    Solution: 919

    We consider the same problem on a smaller scale, using only the numbers from 1 to 11. Let $S' = \{1, 2, 3, \ldots, 11\}$.

    Possible subsets that satisfy the condition are

    $$\{1, 2, 3, 4\}, \quad \{1, 3, 4, 6\},$$

    and so on. These both have 4 elements. It is possible to construct a subset with 5 elements satisfying the condition:

    $$T_1 = \{1, 2, 4, 7, 10\}$$

    A simple proof shows that this is the largest subset we can choose* but it's not necessary to have the proof. Trial and error should be enough to convince you. So 5 is the solution for the smaller problem.

    If we consider the set

    $$T_2 = \{12, 13, 15, 18, 21\}$$

    that is, the set of all elements in T_1 plus 11, by the laws of modular arithmetic it also satisfies the conditions. In fact the set

    $$T_1 \cup T_2 = \{1, 2, 4, 7, 10, 12, 13, 15, 18, 21\}$$

    also satisfies the conditions because $-4 = 7 mod 11$, which means that if there's no difference of 7 or 4 between any elements in T_1, there will also be no difference of 7 or 4 between any element in T_1 and any element in T_2. It then follows that this union is the largest such subset of

    $$S'' = \{1, 2, 3, \ldots, 22\}$$

    satisfying the conditions. We then extend this to

    $$T_3 = T_2 + 11 = \{23, 24, 26, 29, 32\}$$

    and so on. For every set of numbers

    $$\{1, 2, 3, \ldots, 11n\}$$

where n is an integer, the largest subset satisfying the conditions will be

$$T_1 \cup T_2 \cup ...T_n$$

with cardinality $5n$. Noting that $2021 = 2013 + 8 = 183 \times 11 + 8$, the largest subset satisfying the conditions will be

$$\begin{aligned}&(T_1 \cup T_2 \cup ...T_{183}) \cup (T_{184} \cap \{2014, 2015...2021\})\\ &= (T_1 \cup T_2 \cup ...T_{183}) \cup \{2014, 2015, 2017, 2020\}\end{aligned}$$

which has cardinality $183 \times 5 + 4 = 919$

*The proof that there exists no subset of $\{1, 2, ...11\}$ with 6 elements satisfying the conditions is as follows. Partition the set into 6 subsets:

$$\{1, 8\}, \{2, 6\}, \{3, 7\}, \{4, 11\}, \{5, 9\}, \{10\}$$

Note that if we are choosing a subset of $\{1, 2, ...11\}$ with 6 elements such that no two elements differ by 4 or 7, we have to choose one element from each partition. Choosing two from the same partition violates the conditions.

This means 10 has to be in the subset. 10-4 is 6 so 10 being in the subset implies 6 isn't. But that implies 2 is in the subset because it is in the same partition as 6. This chain of logic eventually leads to a contradiction: 10 in implies 6 out implies 2 in implies 9 out implies 5 in implies 1 out implies 8 in implies 4 out implies 11 in implies 7 out implies 3 in implies 10 out. So a satisfactory subset with 6 elements doesn't exist.

D. Proof questions, 10 marks each

11. Let $n \geq 1$. Prove that there exists a set of n^2 distinct real numbers $x_1, x_2, ..., x_{n^2}$ such that every $n \times n$ matrix constructed from these numbers (that is, any $n \times n$ matrix in which each x_i appears as an entry exactly once) is invertible.

    Solution: This can be done in several ways. One class of workable solutions is to choose the elements in such a way that the ratio between any two pairs of elements is never equal. The simplest way to do this is to choose x_i that grow so quickly that the ratio between any two pairs of elements blows up. For example letting $x_1 = 10000$ and each subsequent $x_i = 2^{\prod_{j<i} x_i!}$

12. A group of 2021 friends are at a shopping mall during the COVID-19 pandemic. They are trying their best to maintain social distance, however some of them are standing a bit too close to one other. To solve this problem, each of them sends a text message to the friend standing nearest to them requesting that they move away. No two distances between pairs of friends are equal.

    (a) Prove that at least one of the friends does not receive a text message.

    (b) Prove that none of the friends receives more than 5 text messages.

    Solution. (a) Suppose for contradiction that each friend receives exactly one text message. Then consider the two friends that are standing the minimum distance apart: call them A and B. Each of them must receive a text from the other. If either A or B receives a text message from someone else as well, then by the pigeonhole principle one of the 2021 friends does not receive a text. Hence, we may remove A and B from the rest of the group and repeat this argument. Since 2021 is odd, and we remove a pair at each step, we end up with 1 friend who does not send a text message to any of the 2020 others, which is a contradiction.

    (b) Suppose for contradiction that one of the friends, say A, receives 6 or more messages. Then there must exist 2 of these friends, say B and C, for which the angle $\angle BAC$ is less than or equal to $360/6 = 60°$. Assume WLOG that $AB > AC$. Then, since A receives a text message from B we must have $AB < BC$. However, this would imply $AB^2 < BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos \angle BAC$

    $\Rightarrow 2AB \cdot AC \cos \angle BAC < AC^2$

    $\Rightarrow \cos \angle BAC < \frac{1}{2}\frac{AC}{AB} < \frac{1}{2}$

    This is a contradiction since $\angle BAC < 60°$ implies $\cos \angle BAC > \frac{1}{2}$. Therefore each friend receives at most 5 messages.