WMC 2020 Undergraduate Qualidfying Round Solutions

1. **B** The AM-GM inequality gives $x + y \geq 2\sqrt{xy} = 26$. Equality holds when $x = y = 13$.

2. **D** $\sqrt{0.444\ldots} = \sqrt{\frac{4}{9}} = \frac{2}{3} = 0.666\ldots$.

3. **C** Notice that $(x + \frac{1}{x})^2 = x^2 + \frac{1}{x^2} + 2 \implies 2^2 = x^2 + \frac{1}{x^2} + 2 \implies x^2 + \frac{1}{x^2} = 2$. Repeating this process will give that $x^{1024} + \frac{1}{x^{1024}} = 2$

4. **C** We are integrating over a point and so the result is 0.

5. **C** Integrating by parts (letting $x = u$ and $e^{-2x}dx = dv$) shows that $\int_0^\infty xe^{-2x}dx = \frac{1}{4}$ and so $k = 4$.

6. **A** Let $y = x^x \implies \ln y = x \ln x$. Implicit differentiation gives $\frac{y'}{y} = \ln x + 1$ and so $y' = x^x(\ln x + 1)$

7.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{n-1}}{F_n F_{n+1}} &= \sum_{n=1}^{\infty} \frac{F_{n+1} - F_n}{F_n F_{n+1}} \\
&= \sum_{n=1}^{\infty} (\frac{1}{F_n} - \frac{1}{F_{n+1}}) \\
&= \frac{1}{F_1} - \frac{1}{F_2} + \frac{1}{F_2} - \frac{1}{F_3} + \frac{1}{F_3} - \ldots \\
&= \frac{1}{F_1} \\
&= 1
\end{aligned}$$

8. **E** The chain rule and the fundamental theorem of calculus give that $f'(t) = \frac{3t^2}{1+\ln(t^3)}$ and so $f'(2) = \frac{12}{1+\ln(8)}$.

9. **A**

$$\begin{aligned}
\int_0^{100} \ln(1 + \frac{1}{1 + \lceil x \rceil})dx &= \sum_{n=0}^{99} \int_n^{n+1} \ln(1 + \frac{1}{1 + \lceil x \rceil})dx \\
&= \sum_{n=0}^{99} \int_n^{n+1} \ln(1 + \frac{1}{n+1})dx \\
&= \sum_{n=0}^{99} \ln(1 + \frac{1}{n+1}) \\
&= \sum_{n=0}^{99} \ln(\frac{n+2}{n+1}) \\
&= \ln(\prod_{n=0}^{99} \frac{n+2}{n+1}) \\
&= \ln(101)
\end{aligned}$$

10. **B** The function $f(x) = x^7 e^{-x^2}$ is odd so the integral is 0.

11. **B** To approximate $\sqrt{5}$ we can iterate the formula $a_i - \frac{a_i^2}{2a_i}$ with $a_0 = 2$. We then get $a_2 = \frac{161}{72}$ which happens to be good enough for 4 significant digits. Then $(\frac{1+\sqrt{5}}{2})^4 \approx 6.854$ and so the closest number in the list is 6.86. There are many ways to approximate $\sqrt{5}$.

12. **C** Throughout we will use that if a digit is *unmoved* it must be in its original position.
If the first number is 5 then no value will be *unmoved* and we can arrange the remaining four digits in $4! = 24$ ways.
Now if the second number is 5 then the only way we can have a value that is *unmoved* is if the first number is 1, the rest leave all the numbers *unmoved*. So here, there are $4! - 3! = 18$ ways.
Now suppose 5 is the third number. Then if 1 is the first number, it will be *unmoved*. So 1 cannot be the first number. Now if 1 is the second number we cannot have that $1, 2, 3, 5$ are *unmoved* and so 4 is the only value that can be *unmoved* which cannot happen since 5 would be to the left of 4. So here there are $4! - 3! = 18$ ways.
Now suppose the first number is fourth number is 5. If 1 is the first number, again it is *unmoved* so we omit all such numbers and get $4! - 3! = 18$ numbers. Now if 1 is the second number, then $1, 2, 4, 5$ are *unmoved* and the only way 3 can be *unmoved* is when we have 21354, so we must subtract this case. If the third number is 1 then no value can remain *unmoved*.
So in total we have $24 + 18 + 18 + 18 - 1 = 77$ such rearrangements.

13. **C** For all $k \in \{0, 1, 2, \ldots 9, 10\}$ there are $\binom{10}{k}$ subsets of S with k elements. Now each of those subsets have 2^k elements. So the total number of subsets of subsets of S is $\sum_{k=0}^{10} \binom{10}{k} 2^k = (2+1)^{10} = 3^{10}$.

14. **B** We use coordinates with the careful choice of where we put our circles. Let the circles have centres $A(-\frac{1}{2}; 0)$ $B(\frac{1}{2}; 0)$. Then the equations of the circles are $(x + \frac{1}{2})^2 + y^2 = 1$ and $(x - \frac{1}{2})^2 + y^2 = 1$. Since the square is symmetric about the x axis and the y axis we have set the top right corner of the square to be $P(k; k)$ for some real $k > 0$. Then since k lies on one of these circles, we have $(k + \frac{1}{2})^2 + k^2 = 1$ and so $k = \frac{\sqrt{7}-1}{4} \implies (2k)^2 = \frac{4-\sqrt{7}}{2}$.

15. **C** Writing out the first few terms gives the cycle 20, 101, $\frac{51}{10}$, $\frac{61}{1010}$, $\frac{21}{101}$. So every fifth term is $\frac{21}{101}$.