WMC 2020 Undergraduate Final Round Solutions

Section A

1. $\frac{5\sqrt{2}}{2}$

   We use Cartesian coordinates. Suppose C is the origin i.e. $C(0;0)$. Let $A(4;4)$ and $B(7;0)$, then $E(4;0)$. Now to find D, it lies on the line $y = -x + 7$ an since it lies on a line passing through B and is perpendicular to AC. Also since $\angle C = 45°$, we have that $D(a;a)$ for some real $a > 0$. So $a = -a + 7 \implies a = \frac{7}{2}$. Now the length of DE is $\sqrt{(\frac{7}{2} - 4)^2 + (\frac{7}{2} - 0)^2} = \frac{5\sqrt{2}}{2}$.

2. **540°**

   The heptagram is formed by extending the sides of a regular heptagon with exterior angles equal to $\frac{360°}{7}$. So each angle at the star's tip is equal to $180 - 2 \times \frac{360}{7} = \frac{540°}{7}$ and so their total sum is $540°$.

3. **1**

   If $p = 3$ then $p^3 + p^2 + 11p + 2 = 71$ which is prime. Now suppose that $p \neq 3$ is prime then $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$. If $p \equiv 1 \pmod 3$ then $p^3 + p^2 + 11p + 2 \equiv 1^3 + 1^2 + 11 + 2 = 15 \equiv 0 \pmod 3$ and it cannot be prime. If $p \equiv 2 \pmod 3$ then $p^3 + p^2 + 11p + 2 \equiv 2^3 + 2^2 + 22 + 2 = 36 \equiv 0 \pmod 3$ and so it is divisible by 3 and cannot be prime.

4. **4949**

   Suppose $101 \mid 10^i - 10^j = 10^j(10^{i-j} - 1)$. Since $\gcd(10^k, 101) = 1 \ \forall\ k \in \mathbb{N}$, we must find i and j such that $101 \mid 10^{i-j} - 1$. Considering the powers of 10 starting from 1, we get the following pattern of remainders when divided by 101: $1, 10, -1, -10, 1, 10, -1, -10, \ldots$. So summing the first n remainders adds up to 0 if and only if n is divisible by 4. So it must be if $101 \mid 10^n$ then $4 \mid n$. Then we find all $0 \leq j < i \leq 200$ such that $4 \mid i - j$. There are $197 + 193 + \ldots 5 = 4949$ such pairs.

5. **7**

   Let the number of cards be n and the number of symbols be k. Then by (3), we have that $\binom{k}{2} = \frac{k(k-1)}{2} = 3n \implies n = \frac{k(k-1)}{6}$. Also each symbol must appear exactly $m = \frac{3n}{k}$ times. And so $\binom{n}{2} = \frac{n(n-1)}{2} = k \times \binom{m}{2} = \frac{km(m-1)}{2} = \frac{3n(\frac{3n}{k} - 1)}{2} \implies n(\frac{9}{k} - 1) = 2$. So $(\frac{k(k-1)}{6})(\frac{9}{k} - 1) = 2 \implies k = 3$ or $k = 7$. Now $k = 3$ gives $n = 1$ which cannot be, and $k = 7$ gives $n = 7$. A unique construction exists.

6. $\boldsymbol{\frac{x^2}{2} - \frac{a^2}{2} \ln(x^2 + a^2) + C}$

   $I = \int \frac{x^3}{a^2+x^2} dx = \int (x - \frac{xa^2}{x^2+a^2}) dx = \frac{x^2}{2} - \frac{a^2}{2} \int \frac{1}{u} du$ where $u = x^2 + a^2$. So $I = \frac{x^2}{2} - \frac{a^2}{2} \ln(x^2 + a^2) + C$

7. **0**

   The determinant of a product of matrices is equal to the product of its determinants. So the determinant of the LHS is 0 meaning that the determinant of the RHS is 0. Considering the second row gives $x = 0$.

8. $\mathbf{\frac{\sqrt{2}}{2} - \frac{1}{10}}$

Rationalising denominators gives that the sum is equal to $\sum_{k=2}^{99} \frac{(k+1)\sqrt{k}-k\sqrt{k+1}}{k(k+1)} = \sum_{k=2}^{99} \frac{\sqrt{k}}{k} - \sum_{k=2}^{99} \frac{\sqrt{k+1}}{k+1}$ which telescopes and is equal to $\frac{\sqrt{2}}{2} - \frac{1}{10}$.

9. $\mathbf{\frac{1}{2}}$

Let $I = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x+y+z-w}{x+y+z+w} dxdydzdw$. Now making change of variables and applying Fubini's theorem we have $I = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x+y+w-z}{x+y+z+w} dxdydzdw = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x+w+z-y}{x+y+z+w} dxdydzdw = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{w+z+y-x}{x+y+z+w} dxdydzdw$ and so $4I = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x+y+z-w+x+y+w-z+x+w+z-y+z+y+w-x}{x+y+z+w} dxdydzdw = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{2(x+y+z+w)}{x+y+z+w} dxdydzdw = \int_0^1 \int_0^1 \int_0^1 \int_0^1 2dxdydzdw = 2 \implies I = \frac{1}{2}$.

10. **2**

Consider the linear transformation $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix}$ which compresses the x axis by a factor of 3 and stretches the y axis by a factor of 3. The area is then preserved because the determinant of the matrix is 1. Now essentially the ellipses is rotated by 90° under the transformation and we must reflect the line $2y = x$ about the line $y = x$ and so the new lines is $y = 2x$ and so $m = 2$.

Section B

11.

12. **511**

Let $f^n(x)$ denote the composition of f n times. Notice that $f^{10}(x) = \{1024x\}$. Then $\{1024x\} = x \implies 1024x = k + x$ for some integer k. So $x = \frac{k}{1023}$ and since $x \in [0, 1)$ we have $k = 0, 1, 2, ...1022$ and any such k works. Now the sum of these numbers is $\frac{0+1+\cdots+1022}{1023} = 511$.