2020 Wits Mathematics Competition
Qualifying Round
Undergraduate

Instructions

This exam consists of 15 multiple choice questions. There is one correct answer to each question. There is no penalty for incorrect answers. The first 5 questions are each worth 3 points, the next 5 questions are each worth 4 points and the last 5 questions are each worth 5. The total number of points available is 60. The time limit on this exam is 75 minutes, calculators and geometric implements may NOT be used. If you are using the computer friendly answer sheet you should fill it in in BLACK pen (other colours do not scan well). Time may be given for filling in name, school and other personal details.

"The really unusual day would be one where nothing unusual happens". — Persi Diaconis

A. 3 point questions

1. Two positive real numbers x and y satisfy $xy = 169$. Find the minimal value of $x + y$.
   A. 13
   B. 26
   C. -26
   D. -13
   E. $-\infty$

2. The expression $\sqrt{0.444...}$ is written as a decimal. Find the 100^{th} digit.
   A. 0
   B. 2
   C. 4
   D. 6
   E. 8

3. If $x + \frac{1}{x} = 2$, find the value of $x^{1024} + \frac{1}{x^{1024}}$.
   A. 2^{1024}
   B. 2^{10}
   C. 2
   D. $2^{10} + \frac{1}{2^{10}}$
   E. $\sqrt{2}$

4. Evaluate the integral $\int_1^1 \sqrt{\tan x} dx$.
   A. Undefined
   B. 12
   C. 0
   D. π
   E. 2π

5. For what value of k, if any, will $\int_0^\infty kxe^{-2x}dx = 1$?
   A. $\frac{1}{4}$
   B. 1
   C. 4
   D. 2
   E. No such k exists

B. 4 point questions

6. Compute $\frac{d}{dx}x^x$.
   A. $x^x(\ln x + 1)$
   B. $x^x \ln x$
   C. x^x
   D. $x^x(\ln x - 1)$
   E. $x \ln x$

7. What is the value of
$$\sum_{n=1}^{\infty} \frac{F_{n-1}}{F_n F_{n+1}}$$
where F_n is the n$^{\text{th}}$ Fibonacci number ($F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$).
   A. $\frac{\sqrt{5}-1}{2}$
   B. $\frac{\sqrt{5}+1}{2}$
   C. 1
   D. $\frac{\pi}{2}$
   E. $2\sqrt{2} - 2$

8. If $f(x) = \int_0^{x^3} \frac{dt}{1+\ln t}$, compute $f'(2)$.
   A. $\frac{1}{1+\ln 2}$
   B. $\frac{12}{1+\ln 2}$
   C. $\frac{1}{1+\ln 8}$
   D. $\frac{12}{1+\ln 8}$
   E. $\frac{8}{1+\ln 12}$

9. What is the value of $\int_0^{100} \ln(1 + \frac{1}{\lceil x \rceil})dx$? Where $\lceil x \rceil$ is the smallest integer greater than or equal to x.
   A. $\ln 101$
   B. $\ln 100$
   C. 101
   D. 100
   E. 50.5

10. Compute the integral.

$$\int_{-\pi}^{\pi} x^7 e^{-x^2} dx$$

A. -3

B. 0

C. π^7

D. $7!$

E. $\pi^7 7!$

C. 5 point questions

11. Which of the following numbers is closest to $\left(\frac{1+\sqrt{5}}{2}\right)^4$?

A. $6,55$

B. $6,86$

C. $6,9$

D. $6,72$

E. $6,80$

12. Consider a rearrangement of the values of $\{1, 2, 3, 4, 5\}$. A value is called "unmoved" if it is both smaller than all values to its right and larger than all values to its left.

    For example, the 3 in $\{2, 1, 3, 5, 4\}$ is unmoved as is the 1 in $\{1, 5, 3, 2, 4\}$.

    How many rearrangements of $\{1, 2, 3, 4, 5\}$ have no unmoved values?

    A. 44

    B. 76

    C. 77

    D. 96

    E. 45

13. Let S be a set of 10 elements. We wish to count the number of subsets of subsets of S. More precisely, find the number of pairs (X, Y) such that $X \subseteq Y \subseteq S$?

    A. 0

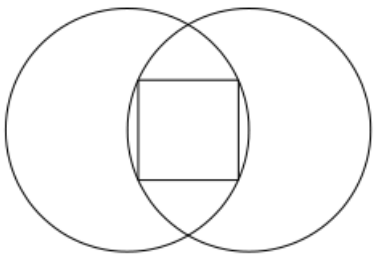
    B. 2^{11}

    C. 3^{10}

    D. 3^{11}

    E. 4^{10}

14. Two circles, each of radius 1, are drawn such that their centres are 1 unit apart. A square is constructed between the two centres such that two of its four vertices lie on the one circle and the other two lie on the other circle as shown. What is the area of the square?



    A. $\frac{3-\sqrt{7}}{4}$

    B. $\frac{4-\sqrt{7}}{2}$

    C. $\frac{\sqrt{7}-1}{2}$

    D. $\frac{3+\sqrt{7}}{4}$

    E. $\frac{4+\sqrt{7}}{2}$

15. Define a sequence (x_n) of real numbers by $x_1 = 20, x_2 = 101$ and

$$x_n = \frac{x_{n-1} + 1}{x_{n-2}}$$

for $n > 2$. What is x_{2020}?

A. 20

B. 101

C. $\frac{21}{101}$

D. $\frac{51}{10}$

E. $\frac{61}{1010}$