WMC 2020 Senior Secondary Final Round Solutions

Section A

1. **7**

   Let x be the number of girls in the classroom and y the total number of learners. Trying all the fractions in the form $\frac{x}{y}$ where $y \le 6$ shows that none of them lie between 40% and 50%. And $40\% < \frac{3}{7} < 50\%$.

2. **1**

   We have $\frac{x^3}{\sqrt{4-x^2}} + x^2 - 4 = 0 \implies x^3 = (4-x^2)\sqrt{4-x^2}$. Then squaring both sides and taking cubic roots gives $x^2 = 4 - x^2 \implies x = \sqrt{2}$ or $x = -\sqrt{2}$ and checking shows that the only solution is $x = \sqrt{2}$.

3. $\mathbf{\frac{5\sqrt{2}}{2}}$

   We use Cartesian coordinates. Suppose C is the origin i.e. $C(0;0)$. Let $A(4;4)$ and $B(7;0)$, then $E(4;0)$. Now to find D, it lies on the line $y = -x + 7$ an since it lies on a line passing through B and is perpendicular to AC. Also since $\angle C = 45°$, we have that $D(a;a)$ for some real $a > 0$. So $a = -a + 7 \implies a = \frac{7}{2}$. Now the length of DE is $\sqrt{(\frac{7}{2} - 4)^2 + (\frac{7}{2} - 0)^2} = \frac{5\sqrt{2}}{2}$.

4. $\mathbf{\frac{3}{2}}$

   Let r be the radius of the circle. Also let O be the origin and let $A(2;\sqrt{5})$. Now let OA meet the circle again at B with $AB = 2r$ since AB is a diameter. Now let C be the foot of the altitude from B on the x axis and let BC intersect the circle again at D. Now since AB is a diameter, we get that $\angle ADB = 90°$ and so Pythagorus on $\triangle ADB$ gives $BD^2 = AB^2 - AD^2 = (2r)^2 - 2^2 \implies BD = \sqrt{4r^2 - 4}$. Now $\triangle BAD \sim \triangle BOC$ so $\frac{AB}{AO} = \frac{BD}{CD} \implies \frac{2r}{3} = \frac{\sqrt{4r^2-4}}{\sqrt{5}}$ and solving gives $r = \frac{3}{2}$.

5. **7**

   Let the number of cards be n and the number of symbols be k. Then by (3), we have that $\binom{k}{2} = \frac{k(k-1)}{2} = 3n \implies n = \frac{k(k-1)}{6}$. Also each symbol must appear exactly $m = \frac{3n}{k}$ times. And so $\binom{n}{2} = \frac{n(n-1)}{2} = k \times \binom{m}{2} = \frac{km(m-1)}{2} = \frac{3n(\frac{3n}{k} - 1)}{2} \implies n(\frac{9}{k} - 1) = 2$. So $(\frac{k(k-1)}{6})(\frac{9}{k} - 1) = 2 \implies k = 3$ or $k = 7$. Now $k = 3$ gives $n = 1$ which cannot be, and $k = 7$ gives $n = 7$. A unique construction exists.

6. **(-1,6) (-5,4) (-4,3) (-2,7)**

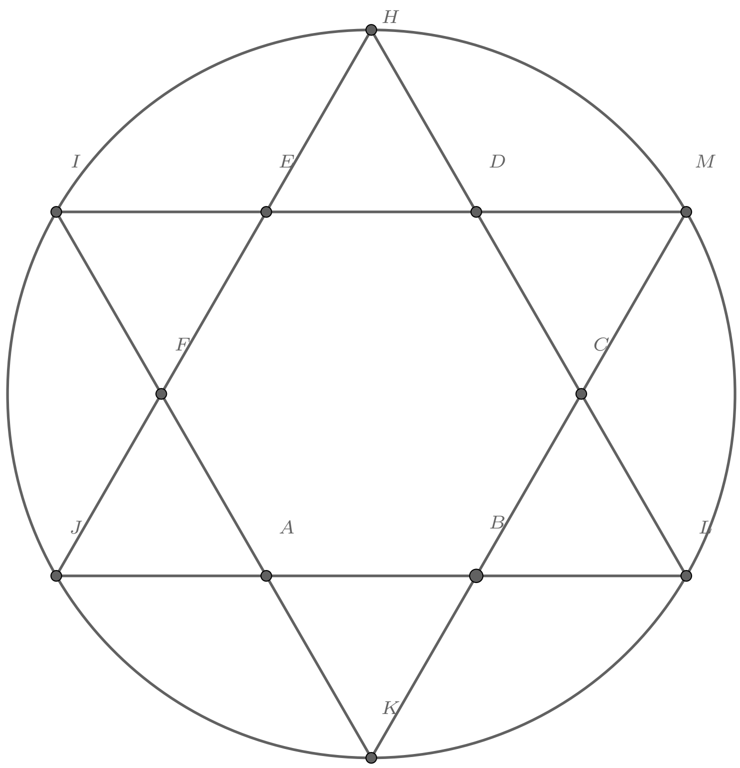
   $xy + 3x - 5y - 17 \iff (x-5)(y+3) = 2$. Now since x and y are integers, then we have four cases: $(x-5, y+3) = (2,1)$, $(x-5, y+3) = (-2,-1)$, $(x-5, y+3) = (-1,-2)$ and $(x-5, y+3) = (1,2)$ which yields the (x,y) pairs $(-1,6)$, $(-5,4)$, $(-4,3)$ and $(-2,7)$.

7. **3**

   Suppose that $a + (a+1) + \cdots + (a + (k-1)) = 369 \implies ka + \frac{k(k-1)}{2} = 369 \implies k(a + \frac{k-1}{2}) = 369$ where a and k are positive integers. Now k has to be a positive factor of 369 so we consider the cases when $k \in \{1, 3, 9, 41, 123, 369\}$. Now $k = 1 \implies a = 369$, $k = 3 \implies a = 122$, $k = 9 \implies a = 37$. Now when $k \ge 41$ we have $\frac{369}{k} \le 9$ but $\frac{k-1}{2} \ge 20$ and so we cannot find a positive a. So there are only three ways.

8. **$1-\frac{\sqrt{3}}{9\pi}$**

Extend the sides of the hexagon. We show that these lines intersect on the circle. The angles of the hexagon are equal to $120°$ and so consider $\triangle HED$ which is equilateral and so is $\triangle HFC$. So the length of the altitude from H onto FC is $\sqrt{3}$ which shows that H is on a circle with radius $\sqrt{3}$.



Now a point P can see exactly two points if and only if it lies outside of the star. The area of the star is $12\times\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{3}$. So the area of the region that is in the circle but not in the star is $3\pi-\frac{\sqrt{3}}{3}$ so the probability that it lies in this region is $\frac{3\pi-\frac{\sqrt{3}}{3}}{3\pi}=1-\frac{\sqrt{3}}{9\pi}$.

9. **3**

If $p=3$ then $p^3+p^2+11p+2=71$ which is prime. Now suppose that $p\neq 3$ is prime then $p\equiv 1 \pmod 3$ or $p\equiv 2 \pmod 3$. If $p\equiv 1 \pmod 3$ then $p^3+p^2+11p+2\equiv 1^3+1^2+11+2=15\equiv 0 \pmod 3$ and it cannot be prime. If $p\equiv 2 \pmod 3$ then $p^3+p^2+11p+2\equiv 2^3+2^2+22+2=36\equiv 0 \pmod 3$ and so it is divisible by 3 and cannot be prime.

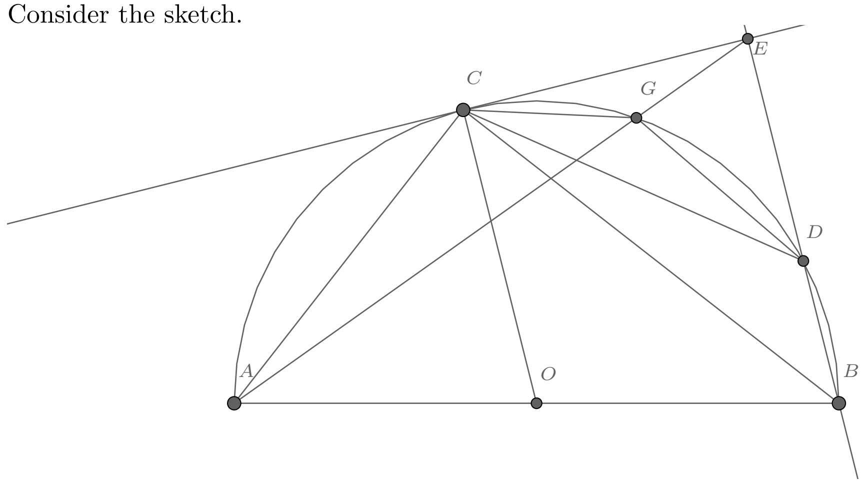
10. **999**

$3^3+5^3+\dots 1999^3=(1^3+2^3+\cdots+1999^3)-(2^3+4^3+\cdots+1998^3)-1^3=(\frac{1999\times 2000}{2})^2-2^3(\frac{999\times 1000}{2})^2-1=1999000^2-2(999000)^2-1=(2\times 999000+1000)^2-2(99900)^2-1$. Now $(2\times 999000+1000)^2-2(99900)^2-1\equiv 100^2-1=999\times 1001=999000+999\equiv 999 \pmod{999000}$.

Section B

11. $15x^2 - 7y^2 = 9 \implies 7y^2 = 15x^2 - 9$ and since the RHS is divisible by 3, we have that $3 \mid 7y^2 \implies 3 \mid y$. Let $y = 3a$ for some $a \in \mathbb{Z}$. Then $15x^2 - 7(3a)^2 = 9 \implies 5x^2 - 21a^2 = 3$ and so $3 \mid 5x^2 \implies 3 \mid x^2$ so let $x = 3b$ for $b \in \mathbb{Z}$. Then $5(3b)^2 - 21a^2 = 9 \implies 15b^2 - 7a^2 = 1$. Now $7a^2 = 15b^2 - 1 \equiv -1 \pmod 3$ or equivalently $a^2 \equiv 2 \pmod 3$ which is impossible. So there cannot exist such x and y.

12. Consider the sketch.



We have $\angle ABC = \angle DBC$ since $AC = CD$ and $\angle AOC = 90°$ since AB is a diameter. Also $\angle AOC = 2 \times \angle ABC$ and so $\angle AOC = \angle ABD \implies CO \parallel BE$. Now since $CDBA$ is cyclic, we have that $\angle DGE = \angle DBA$ and so $\triangle EGD \sim \triangle EBA \implies \frac{GD}{AB} = \frac{EG}{EB}$ (1). By tan-chord $\angle ECG = \angle EAC \implies \triangle CEG \sim \triangle AEC \implies \frac{AC}{CG} = \frac{CE}{EG}$ (2).
Multiplying (1) and (2) gives $\frac{AC}{CG} \times \frac{GD}{AB} = \frac{EG}{EB} \times \frac{EC}{EG} \implies \frac{GD}{CG} \times \frac{AC}{AB} = \frac{EC}{EB}$. Also since $\triangle CEB \sim \triangle ACB$ we have $\frac{CE}{BE} = \frac{AC}{CB}$. And so $\frac{GD}{CG} = \frac{AB}{AC} \times \frac{AC}{BC} = \frac{AB}{AC}$ and $AB > BC$ since AC is the hypotenuse of $\triangle ABC$.