WMC 2018 Senior Secondary Solutions

Section A

1. **D** $(2016 - 2017) + (2018 - 2019) + 2020 = -1 - 1 + 2020 = 2018$

2. **D** $\frac{6a+18b}{12a+6b} = \frac{a+3b}{2a+b} = \frac{\frac{1}{2}+\frac{9}{2}}{1+\frac{3}{2}} = 10.$

3. **C** $0 < \frac{2}{7} < 1 < 7 < \frac{31}{4} < 8$ so the only integers are $1, 2, 3, 4, 5, 6, 7$.

4. **C** There are $2016 \div 4 = 504$ groups of $1, 2, 3, 4$ and an extra 1 and 2. Thus the total sum is is $504(1 + 2 + 3 + 4) + 1 + 2 = 5043$.

5. **E** $P^{2n}Q^m = (2^m)^{2n}(3^n)^m = 4^{mn} \times 3^{mn} = 12^{mn}$

6. **E** Since $y < 20$ we have $x \leq 2$ and since x and y are distinct, we have $x = 2$ and $y = 8$ and so $y^2 = 64$.

7. **D** Let k be the number of peanuts that John ate on the final night. Then $k + (k - 6) + (k - 12) + (k - 18) = 120 \implies k = 39$.

8. **D** The sum of the numbers from 1 to 11 is $\frac{11\times 12}{2} = 66$ and the sum of the remaining ten numbers must be 61 so the number that must be removed is $66 - 61 = 5$

9. **D** Let x be the age of the youngest child. Then $x + (x + 1) + (x + 2) = 42 \implies x = 13$. The age of the oldest child is $13 + 6 = 19$.

10. **C** $\frac{40}{100} \times P = \frac{10}{100} \times Q \implies 4P = Q$ and so P is 25% of Q.

Section B

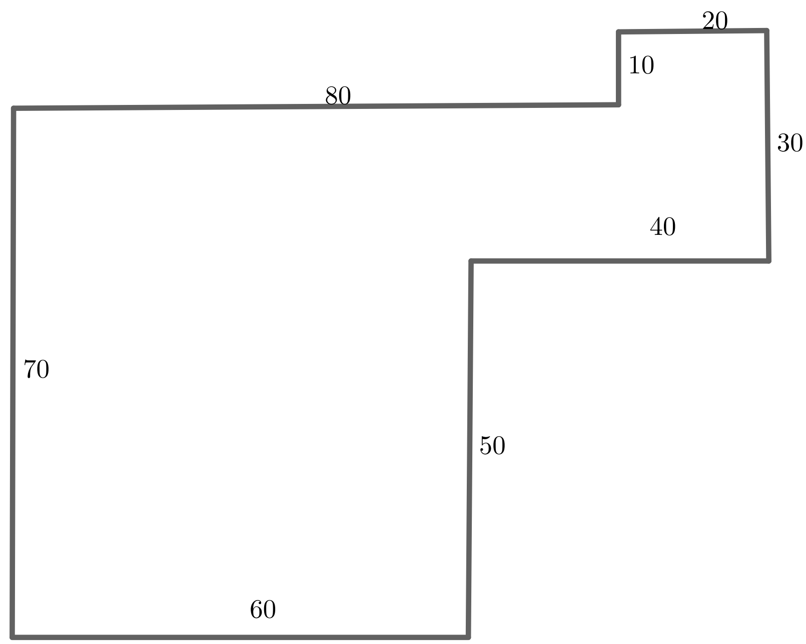
11. **6** Pythagorus on $\triangle ABD$ and $\triangle CBD$ gives $AD = \sqrt{x^2 + 16}$ and $CD = \sqrt{x^2 + 81}$. Then Pythagorus on $\triangle ACD$ gives $x^2 + 16 + x^2 + 81 = 13^2 \implies x = 6$.

12. $\boldsymbol{\frac{2x^2}{5}}$ Let s be the width of the rectangle and the length is $2s$. Then the area of the rectangle is $2s^2$. Now we have that $s^2 + (2s)^2 = x^2$ and so $s^2 = \frac{x^2}{5} \implies$ Area of rectangle $= 2s^2 = \frac{2x^2}{5}$.

13. **26** Bertha has $30 - 6 = 24$ granddaughters and so 4 of her 6 daughters have 6 daughters. So $30 - 4 = 26$ of her daughters and granddaughters do not have daughters.

14. **196** Consider the sets $\{1, 2, 3\}$ and $\{6, 7, 8\}$. We must choose at least element from both of these sets, which we can do in $2^3 - 1 = 7$ ways. Now for the set $\{4, 5\}$ we can choose any (including the empty set) subset so we have $2^2 = 4$ choices. So the total number of such subsets is $7 \times 7 \times 4 = 196$.

15. **5200** The polygon can be uniquely formed as shown in the diagram. The area is then equal to $80 \times 100 - (10 \times 80 + 50 \times 40) = 5200$.



16. $\mathbf{\frac{1}{10}}$ Let $|BD| = a \implies |AB| = 4a$ and $|CD| = b \implies |AC| = 9a$. Then $|BC| = |BD - CD| = a - b$. Also $AD = 5a = 10b \implies a = 2b \implies |BC| = 2b - b = b$. So $\frac{|BC|}{|AD|} = \frac{b}{10b} = \frac{1}{10}$.

17. **50** Suppose without loss of generality that we have a 100 kg orange. Then 80 kg is water, and 20 kg is other content. Now 75% of 80 is 60 so we have $80 - 60 = 20$ kg of water left. Then the percentage of water in the remaining orange is $20 \div (20 + 20) \times 100 = 50\%$.

18. **222** Note that 2018 in base 3 is 2202202. So consider $k = 1, 2, 3, 4, 5, 6$. Then there are 2^k trinary numbers of length k that satisfy the condition since each of the digits can be 1 or 2. Then suppose the trinary number has seven digits. Then if the first digit is 1 the rest of the numbers can be any of one and two so we get $2^6 = 64$ numbers, and if the first digit is 2, the next has to be 1 or it is too large, and so the rest of the 5 can be filled in $2^5 = 32$ ways. So in total we have $2+4+8+16+32+64+64+32 =$ numbers.

19. **13** It is clear that 60! and 65! end in more zeros than 55! so we must find how many zeros 55! ends in. This can be done by counting how many factors of 5 are in the product $55 \times 54 \times \cdots \times 2 \times 1$ since to get a zero, we need a factor of 10 made by a 2 and a 5 and there are more $2's$ than $5's$. Now there are 11 numbers in the product divisible by 5 and 2 are divisible by 25, so in total we have $11 + 2 = 13$ factors of 5. Thus there are 13 zeros at the end of the sum.

20. $\mathbf{\frac{2^{p-1}}{2^{p-1}-1}}$ Since $p > 1$, the numerator and denominator converge. Let $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots = L$. Now $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots - 2(\frac{1}{2^p} + \frac{1}{4^p} + \ldots) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots - \frac{2}{2^p}(\frac{1}{2^p} + \frac{1}{2^p} + \ldots)$
$= L - \frac{1}{2^{p-1}}(L) = \frac{L(2^{p-1}-1)}{2^{p-1}}$
Then $\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\ldots} = L \times \frac{2^{p-1}}{L(2^{p-1}-1)} = \frac{2^{p-1}}{2^{p-1}-1}$.

Section C

Question 21

No, it is not possible.
There are 2^{10} different bulb configurations. Now there are also 36 5×5 sub-grids and 64 3×3 sub-grids. Now suppose we choose some of these sub-grids: the number of ways is $2^{64} \times 2^{36} = 2^{100}$. Now label the 1×1 grids by (i, j) where $1 \leq i, j \leq 10$ with the top left square being $(1, 1)$. Choose the 5×5 grids with top left squares $(1, 1)$, $(1, 4)$, $(4, 1)$, $(4, 4)$ and the 3×3 grids with top left corners $(1, 1)$, $(1, 6)$, $(6, 1)$, $(6, 6)$.This leaves the configuration unchanged. So there exists a configuration of the bulbs that can be reached by two different grid choices, hence there must be one that cannot be reached.

Question 22

The only prime in the sequence is 101. Clearly 101 is prime. We then show that there are no other primes in the sequence.
The nth term in the sequence is equal to $\sum_{k=0}^{n} 10^{2k} = \frac{(10^2)^{n+1}}{99} = \frac{(10^{n+1}-1)(10^{n+1}+1)}{99}$. Now $10^{n+1} - 1$ is divisible by 9 and let $\frac{10^{n+1}-1}{9} = K_n \in \mathbb{N}$. So the nth term in the sequence is $K_n \times \frac{10^{n+1}+1}{11}$ which we know is a positive integer, and since 11 is prime, it must divide either K_n or $10^{n+1} - 1$. Now since $n \geq 2$ both K_n and $10^{n+1} - 1$ are greater than 11 and so if 11 divides one of them, it is in the form $11a$ for some positive integer a greater than 1. Then the nth term is the product of two numbers greater that 1 and cannot be prime.