WMC 2020 Junior Secondary Final Round Solutions

Section A

1. **51**

   $204 = 2^2 \times 3 \times 17$. Now any perfect square is the product of distinct primes with even powers. So we need to at least multiply 204 by $3 \times 17 = 51$ in order to get a perfect square.

2. **405**

   Notice that $n \geq 400$ since otherwise $2^{400} \times 5^n \leq 10^{400}$ and so it has most 400 digits. Now $2^{400} \times 5^n = 10^{400} \times 5^{n-400}$. Now this will be the number 5^{400-n} followed by 400 zeros. So we must find the smallest power of 5 with at least 4 digits which is 5^5 and so $n - 400 = 5 \implies n = 405$.

3. **37**

   This is equivalent to finding the number of perfect squares less than or equal to 1400.
   Now $37^2 = 1369 \leq 1400 < 1444 = 38^2$ and so there are 37 perfect squares less than or equal to 1400.

4. **10:50**

   Suppose the trains meet x minutes after the second train leaves L. Also let the first train move at a constant speed of $s\ metres/min$ and the second train move at $t\ metres/min$. Then after these x minutes, the first train has moved $s(30+x)$ metres and the first train has moved tx metres. Now since the first train arrives at L 40 minutes after the trains pass each other we have $40 \times s = tx$ and since the second train arrives at K 100 minutes after the trains pass each other we have $100 \times t = s(30+x)$. Now multiplying these two equations gives $4000st = stx(30+x) \implies x^2+30x-4000 = 0 \implies (x+80)(x-50) = 0 \implies x = 50$. So the trains meet at $10:50$.

5. **7**

   Let x be the number of girls in the classroom and y the total number of learners. Trying all the fractions in the form $\frac{x}{y}$ where $y \leq 6$ shows that none of them lie between 40% and 50%. And $40\% < \frac{3}{7} < 50\%$

6. **6210001000**

   We do trial and error on the first digit. We start when it is as large as possible since the larger cases are simpler. If the first digit is 9 then all the other digits must be 0 and so the number cannot be autobiographical. If the first digit is 8 then the $9th$ digit must be 1 and the rest of the digits must be 0 which is not an autobiographical number. When the first digit is 7 the $8th$ digit must be at least 1 and 7 of the remaining 8 digits must be 0. This leaves that the only possibility is 71000001000 which is not an autobiographical number. Now if we try the first digit to be 6 we get 6210001000 which is valid.

7. **3**

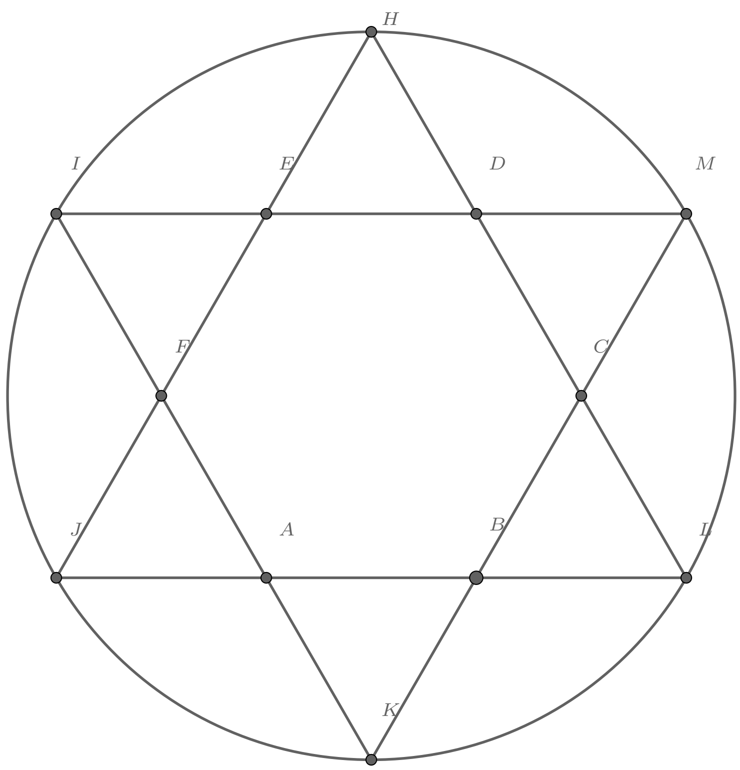
   Suppose that $a + (a+1) + \cdots + (a+(k-1)) = 369 \implies ka + \frac{k(k-1)}{2} = 369 \implies k(a + \frac{k-1}{2}) = 369$ where a and k are positive integers. Now k has to be a positive factor of 369 so we consider the cases when $k \in \{1, 3, 9, 41, 123, 369\}$. Now $k = 1 \implies a = 369$, $k = 3 \implies a = 122$, $k = 9 \implies a = 37$. Now when $k \geq 41$ we have $\frac{369}{k} \leq 9$ but $\frac{k-1}{2} \geq 20$ and so we cannot find a positive a. So there are only three ways.

8. **7**

Let the number of cards be n and the number of symbols be k. Then by (3), we have that $\binom{k}{2} = \frac{k(k-1)}{2} = 3n \implies n = \frac{k(k-1)}{6}$. Also each symbol must appear exactly $m = \frac{3n}{k}$ times. And so $\binom{n}{2} = \frac{n(n-1)}{2} = k \times \binom{m}{2} = \frac{km(m-1)}{2} = \frac{3n(\frac{3n}{k}-1)}{2} \implies n(\frac{9}{k} - 1) = 2$. So $(\frac{k(k-1)}{6})(\frac{9}{k} - 1) = 2 \implies k = 3$ or $k = 7$. Now $k = 3$ gives $n = 1$ which cannot be, and $k = 7$ gives $n = 7$. A unique construction exists.

9. $\mathbf{1 - \frac{\sqrt{3}}{9\pi}}$

Extend the sides of the hexagon. We show that these lines intersect on the circle. The angles of the hexagon are equal to $120°$ and so consider $\triangle HED$ which is equilateral and so is $\triangle HFC$. So the length of the altitude from H onto FC is $\sqrt{3}$ which shows that H is on a circle with radius $\sqrt{3}$.



Now a point P can see exactly two points if and only if it lies outside of the star. The area of the star is $12 \times \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}$. So the area of the region that is in the circle but not in the star is $3\pi - \frac{\sqrt{3}}{3}$ so the probability that it lies in this region is $\frac{3\pi - \frac{\sqrt{3}}{3}}{3\pi} = 1 - \frac{\sqrt{3}}{9\pi}$.

10. $\mathbf{\frac{5\sqrt{2}}{2}}$

We use Cartesian coordinates. Suppose C is the origin i.e. $C(0; 0)$. Let $A(4; 4)$ and $B(7; 0)$, then $E(4; 0)$.

Now to find D, it lies on the line $y = -x + 7$ an since it lies on a line passing through B and is perpendicular to AC. Also since $\angle C = 45^\circ$, we have that $D(a; a)$ for some real $a > 0$. So $a = -a + 7 \implies a = \frac{7}{2}$. Now the length of DE is $\sqrt{(\frac{7}{2} - 4)^2 + (\frac{7}{2} - 0)^2} = \frac{5\sqrt{2}}{2}$.

Section B

11. We have that $\angle FBC = \angle EDC = 90^\circ - 60^\circ = 30^\circ$. Since $DE = AD = CD$, triangle CED is isosceles. Then $\angle DCE = \frac{180^\circ - \angle CDE}{2} = 75^\circ \implies \angle ECB = 15^\circ$. Similarly, we have $\angle FCD = 15^\circ$ and so $\angle ECF = 60^\circ$. Also since $\triangle DEC \equiv BFC$ we have that $CE = FC$ and so $\triangle EFC$ is isosceles with one angle equal to $60^\circ \implies \triangle EFC$ is equilateral.

12. We will show that 6 has to divide xyz.
Suppose that x, y and z are all odd. Then so are x^3, y^3 and z^3. Then $x^3 + y^3 + z^3$ is odd and cannot be divisible by 18. So at least one of x,y and z is even and so 2 divides xyz. Again, suppose that x, y and z are all not divisible by 3. Then for any $a \in \{x, y, z\}$ we have $a \equiv 1, 2, 4, 5, 7, 8 \pmod 9$. Now taking each of these and observing the remainder when divided by 9 shows that $a^3 \equiv 1$ or $2 \pmod 9$. So we cannot have $x^3 + y^3 + z^3 \equiv 0 \pmod 9$ and so 18 cannot divide $x^3 + y^3 + z^3$. Then at least of of x, y, z must be divisible by 3 and so 3 divides xyz. Therefore $6 \mid xyz$.