MATHEMATICAL INDUCTION
for the
OLYMPIAD ENTHUSIAST

David Jacobs

INTRODUCTION

The South African Mathematical Society has the responsibility for selecting and training teams to represent South Africa in the annual International Mathematical Olympiad (IMO).

The process of finding a team to go to the IMO is a long one. It begins with a nationwide Talent Search, in which students are sent sets of problems to solve. Their submissions are marked and returned with comments, full solutions and a further set of problems. The principle behind the Talent Search is straightforward: the more problems you solve, the higher up the ladder you climb and the closer you get to selection.

The best students in the Talent Search are invited to attend Mathematical Camps in which specialised problem-solving skills are taught. The students also write a series of challenging Olympiad-level problem papers, leading to selection of a team of six to go to the IMO.

The booklets in this series cover topics of particular relevance to Mathematical Olympiads. Though their primary purpose is preparing students for the International Mathematical Olympiad, they can with profit be read by all interested high school students who would like to extend their mathematical horizons beyond the confines of the school syllabus. They can also be used by teachers and university mathematicians who are interested in setting up Olympiad training programmes and need ideas on topics to cover and sample Olympiad problems.

Titles in the series published to date are

1. *The Pigeon-hole Principle*, by Valentin Goranko
2. *Topics in Number Theory*, by Valentin Goranko
3. *Inequalities for the Olympiad Enthusiast*, by Graeme West
4. *Graph Theory for the Olympiad Enthusiast*, by Graeme West
5. *Functional Equations for the Olympiad Enthusiast*, by Graeme West
6. *Mathematical Induction for the Olympiad Enthusiast*, by David Jacobs

Details of the South African Mathematical Society's Mathematical Talent Search may be obtained by writing to

Mathematical Talent Search
Department of Mathematics and Applied Mathematics
University of Cape Town
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The International Mathematical Olympiad Talent Search is sponsored by the Old Mutual.

J H Webb
May 1996

Mathematical Induction for the Olympiad Enthusiast

David Jacobs

Mathematical induction is a simple but powerful reasoning tool that is useful in solving a wide variety of problems: arithmetical, algebraic and geometric. The standard school textbook approach to this topic has been the rather limited use of mathematical induction to prove number-theoretic statements like inequalities and formulae for series. The approach of most university textbooks is to discuss the principle of mathematical induction as part of an axiomatic system for reasoning with the natural numbers, focusing on the equivalence of this principle with other number theoretic statements. For the Olympiad enthusiast, or any school student looking for some mathematical enrichment, neither of these approaches is particularly satisfactory, the former being too elementary and limited, the latter being too abstract and technical.

In the light of this, my approach is to introduce and discuss mathematical induction largely in the context of problem-solving, and in particular to discuss problems of the type that appear regularly in mathematics competitions. In keeping with this approach, the first section of this booklet starts with a particular problem, and extracts from its solution some of the basic essence of the principle of mathematical induction. The second section introduces mathematical induction proper, and contains some of the more routine problems in the subject. The third section discusses some of the variants (in fact, most are actually logical equivalents) of the principle of mathematical induction, again opting for a problem-solving rather than a theoretical treatment, in keeping with the style of this booklet. The last section focuses on some harder problems, where it is less obvious to see exactly where and how the principle is used.

1. The Tromino Problem

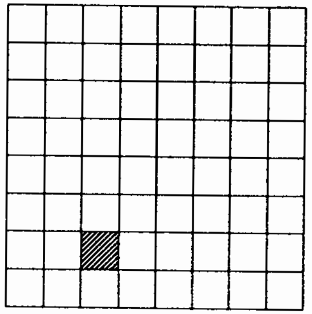


FIGURE 1

Suppose you have an 8×8 grid of squares as shown in Figure 1. Pick an arbitrary square and shade it black. You are left with 63 squares. We now pose the following question: Is it possible to cover the remaining 63 squares with 21 shapes of the form given in Figure 2 (we shall call these shapes *trominoes*), with all remaining squares covered and no overlaps?

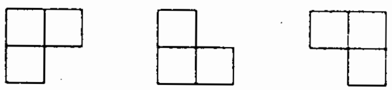


FIGURE 2

To solve this problem, first consider a 2×2 grid, and shade an arbitrary square black. We are then left with 3 remaining squares which form a tromino. Hence it is always possible to cover a 2×2 grid with one square shaded with one tromino (see Figure 3).

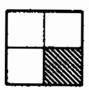


FIGURE 3

Next consider a 4×4 grid with one square shaded. We can divide this grid into four 2×2 grids as shown in Figure 4. One of these contains a shaded square, and hence this grid can be covered by a tromino. Now place a tromino in the centre of the 4×4 grid as shown in Figure 5. Notice that the three remaining 2×2 grids each have exactly one square covered, and hence can be covered by trominoes as in Figure 6. Hence we have solved the 4×4 case.

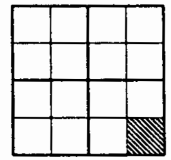


FIGURE 4

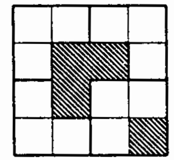


FIGURE 5

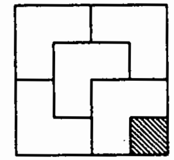


FIGURE 6

We use the same trick for the 8×8 grid: Divide it into four 4×4 grids, and position a tromino in the centre in such a way that each of four grids have one square covered (see Figure 7). We can now use our procedure for the 4×4 grid to cover each of the four grids to yield the solution shown in Figure 8.

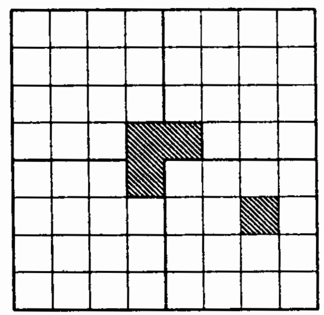


FIGURE 7

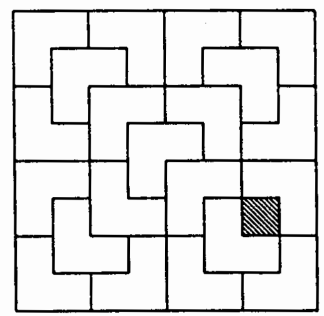


FIGURE 8

Now that we have a solution, let us reflect a bit on some of the essence of the problem and its solution. Notice that this problem conceals a more general problem: It is possible to extend this procedure to a 16×16

grid, or in fact, to any grid with dimensions $2^n \times 2^n$, where $n \in \mathbb{N}$. ($\mathbb{N}$ is the set of natural numbers, that is, the set $\{1, 2, 3, \ldots\}$).

Let $P(n)$ be the statement: "For every $2^n \times 2^n$ square grid with an arbitrary square shaded, it is possible to cover the rest of the squares with trominoes." Notice that the statement $P(n)$ depends on n, a natural number. What we have shown is that $P(n)$ is true for all $n \in \mathbb{N}$. We did this as a two step process:

(i) We showed $P(1)$ is true, that is, we could solve the 2×2 case.

(ii) We showed, that if $P(k)$ was true for some $k \in \mathbb{N}$, then $P(k+1)$ was true. In our particular case, we showed that if we could solve the problem for a $2^k \times 2^k$ grid, we could solve the problem for a $2^{k+1} \times 2^{k+1}$ grid by dividing it up into four $2^k \times 2^k$ grids.

Using these assumptions, note that since $P(1)$ is true, $P(2)$ would be true by (ii). Now, since $P(2)$ is true, by (ii) we have $P(3)$ is true and so on to conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

It is essentially this two-step process that forms the heart of any proof by mathematical induction. We will formalize this method in the next section, but first try to tackle the following exercise to get the feel of this method of proof.

EXERCISE 1.

Given an $n \times n$ grid ($n \geq 3$) with an arbitrary square shaded, show that it is possible to cover up the remaining squares using only 3×1 and 2×1 rectangles.

2. Mathematical Induction Introduced

We can now formally state the Principle of Mathematical Induction.

THE PRINCIPLE OF MATHEMATICAL INDUCTION (PMI). *Consider an infinite sequence of mathematical propositions*

$$P(1), P(2), P(3), \ldots$$

Suppose that

(1) $P(1)$ *is true (that is, can be proved),*

(2) *For each* $m \in \mathbb{N}$, $P(m) \implies P(m+1)$ *(that is, we can prove* $P(m+1)$ *on the assumption that* $P(m)$ *is true).*

Then all the propositions of the sequence must be true, that is, $P(n)$ *is true for all* $n \in \mathbb{N}$.

Notice that we regard the PMI as an axiom, that is, we accept it as valid without proof. In fact, the PMI (or one of its many equivalents) is regarded as one of the basic axioms in any workable mathematical theory that involves the natural numbers.

We now give an example of a standard result proved using the PMI.

EXAMPLE 2.1. *Prove that* $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$ *for each* $n \in \mathbb{N}$.

SOLUTION. For each $n \in \mathbb{N}$, let $P(n)$ be the statement

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$P(1)$ is true since $1^2 = \dfrac{1 \times 2 \times 3}{6}$. Now suppose that for some $m \in \mathbb{N}$, $P(m)$ is true, that is

$$1^2 + 2^2 + 3^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6}.$$

Now

$$
\begin{aligned}
&1^2+2^2+3^2+\cdots+m^2+(m+1)^2\\
&=\frac{m(m+1)(2m+1)}{6}+(m+1)^2\\
&\quad\text{since } P(m) \text{ is true}\\
&=\frac{2m^3+3m^2+m+6m^2+12m+6}{6}\\
&=\frac{2m^3+9m^2+13m+6}{6}\\
&=\frac{(m+1)(m+2)(2m+3)}{6}\\
&=\frac{[m+1]([m+1]+1)(2[m+1]+1)}{6}.
\end{aligned}
$$

Thus $P(m+1)$ is true. Hence by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$, which is precisely the statement we wish to prove. □

When using the PMI, we call the proof for step (1) [$P(1)$ is true] the *base case*, and the proof for step (2) [$P(m) \implies P(m+1)$] the *induction step*. In the induction step $P(m)$, the statement we assume true, is called the *induction hypothesis*. The variable on which the statement P depends (in our case n) is called the *induction parameter*.

Sometimes the selection of a statement $P(n)$ is not all that obvious, as the following example shows.

EXAMPLE 2.2. *Show that, if x is a positive real number, then $(1+x)^n \geq nx$ for all $n \in \mathbb{N}$.*

SOLUTION. Let us first try putting $P(n)$ to be the statement "$(1+x)^n \geq nx$". $P(1)$ is true since $1+x \geq x$. Now suppose for some $m \in \mathbb{N}$, $P(m)$ is true. Then

$$
\begin{aligned}
(1+x)^{m+1} &= (1+x)^m(1+x)\\
&\geq mx(1+x) \quad [P(m) \text{ is true}]\\
&= mx+mx^2.
\end{aligned}
$$

But, since $mx+mx^2 \not\geq (m+1)x$ (take $m=1$ and $x=0.5$ for example), we cannot conclude $P(m+1)$ is true. Hence we are stuck!

A less obvious approach is to let $Q(n)$ be the stronger statement "$(1+x)^n \geq nx+1$". Again $Q(1)$ is true since $1+x \geq x+1$. Suppose for some $m \in \mathbb{N}$, $Q(m)$ is true. Then

$$
\begin{aligned}
(1+x)^{m+1} &= (1+x)^m(1+x)\\
&\geq (mx+1)(1+x) \quad [P(m) \text{ is true}]\\
&= mx^2+(m+1)x+1\\
&\geq (m+1)x+1.
\end{aligned}
$$

Hence $Q(m+1)$ is true, and by the PMI, $Q(n)$ is true for all $n \in \mathbb{N}$. To complete the problem, one need only notice that $nx+1 > nx$ for all $n \in \mathbb{N}$. Thus $P(n)$ is true for all $n \in \mathbb{N}$. □

EXERCISES 2.

(1) Show that $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for all $n \in \mathbb{N}$.

(2) Show that $1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$ for all $n \in \mathbb{N}$.

(3) Show that $2^{n+4} \geq (n+4)^2$ for all $n \in \mathbb{N}$.

(4) Show that a^n-b^n is divisible by $a-b$ for all $a,b,n \in \mathbb{N}$ with $a \neq b$.

(5) The Fibonacci sequence

$$F_1=1, F_2=1, F_3=2, F_4=3, F_5=5, F_6=8, \ldots$$

is defined as follows: $F_1=F_2=1$ and $F_n=F_{n-2}+F_{n-1}$ for all $n \geq 3$. Prove that the following statements are true for all $n \in \mathbb{N}$.

(a) $F_1+F_2+\cdots+F_n=F_{n+2}-1$.

(b) $|F_{n+2}F_n-(F_{n+1})^2|=1$.

(c) 5 divides n if and only if 5 divides F_n.

(6) Prove that for all $n \in \mathbb{N}$,

$$\frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n} \leq \frac{1}{\sqrt{2n}}.$$

(7) [**Mathematical Induction**, *Slinko*] Show that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

for all $n \in \mathbb{N}$.

3. Some Variations on Mathematical Induction

In this section, we look at a range of variations on the basic principle of mathematical induction. In all cases we will give a proof that the altered principle can be derived from the PMI.

The first variation is to alter the base case (i.e. $P(1)$). This can be formally stated as follows:

THE PRINCIPLE OF MATHEMATICAL INDUCTION - VARIANT 1 (PMI1). *Consider an infinite sequence of mathematical propositions*

$$P(1), P(2), P(3), \ldots$$

Let $k \in \mathbb{N}$. Suppose that

(1) *$P(k)$ is true (that is, can be proved),*
(2) *For each $m \in \mathbb{N}, m \geq k$, $P(m) \implies P(m+1)$ (that is, we can prove $P(m+1)$ on the assumption that $P(m)$ is true).*

Then $P(n)$ is true for all $n \in \mathbb{N}, n \geq k$.

Proof. For each $n \in \mathbb{N}$, let $Q(n) = P(n+k-1)$. A check shows that the sequence $Q(1), Q(2), Q(3), \ldots$ satisfies the conditions of the original PMI, and thus $Q(n)$ is true for all $n \in \mathbb{N}$, and hence $P(n)$ is true for all $n \geq k$. □

[*A word of warning*: The abbreviations PMI and PMI1 for the various principles of mathematical induction are not standard, and should not be used in mathematics competitions.]

EXAMPLE 3.1. *Show that $3^n > n^3$ for all $n \in \mathbb{N}$, $n \geq 4$.*

SOLUTION. The result is true for $n = 4$ since $81 > 64$. Now suppose, for some $m \geq 4$, $3^m > m^3$. Then

(a) $$3^{m+1} = 3(3^m) > 3m^3.$$

Also, since $m > 3$, we have

(b) $$m^3 > 3m^2.$$

Again, using $m > 3$, we have $m(m-3) > 1$ and hence

(c) $$m^3 \geq m^2 > 3m + 1.$$

Thus

$$3m^3 = m^3 + m^3 + m^3 > m^3 + 3m^2 + 3m + 1 = (m+1)^3.$$

using (b) and (c), and hence by (a),

$$3^{m+1} > (m+1)^3.$$

Thus by the PMI1, $3^n > n^3$ for all $n \in \mathbb{N}$, $n \geq 4$. □

The PMI can be generalized even further.

THE PRINCIPLE OF MATHEMATICAL INDUCTION - VARIANT 2 (PMI2). *Consider an infinite sequence of mathematical propositions*

$$P(1), P(2), P(3), \ldots$$

Let $a, d \in \mathbb{N}$. Suppose that

(1) *$P(a)$ is true (that is, can be proved),*
(2) *For each $m \in \mathbb{N}$, $P(a + (m-1)d) \implies P(a + md)$ (that is, we can prove $P(a + md)$ on the assumption that $P(a + (m-1)d)$ is true).*

Then $P(a+(n-1)d)$ is true for all $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, let $Q(n) = P(a+(n-1)d)$. A check shows that the sequence $Q(1), Q(2), Q(3), \ldots$ satisfies the conditions of the original PMI, and thus $Q(n)$ is true for all $n \in \mathbb{N}$, and hence $P(a+(n-1)d)$ is true for all $n \in \mathbb{N}$. $\square$

One use, for example, of the PMI2 is to prove a statement $P(n)$ true for all even natural numbers, using $a = 2$ and $d = 2$. In order to establish the result, we would have to prove that $P(2)$ is true, and for any natural number m, the truth of $P(2m)$ implies the truth $P(2m+2)$.

EXAMPLE 3.2. *[IMO 1979] A convex polygon with an even number of sides, all equal to each other and with pairs of opposite sides parallel is called a* parpolygon . *Prove that any parpolygon can be dissected into rhombi.*

SOLUTION. Let $P(n)$ be the statement "An n-sided parpolygon can be dissected into rhombi". We will use PMI2 with $a = 4$ and $d = 2$. $P(4)$ is true since a 4-sided parpolygon is automatically a rhombus. Now suppose for some $m \in \mathbb{N}$, we have that $P(2m+2)$ is true. Consider a $(2m+4)$-sided parpolygon with vertices $A_1, A_2, \ldots, A_{2m+4}$. Construct points $B_{m+2}, B_{m+3}, \ldots, B_{2m+2}$ inside the parpolygon such that

$$A_{m+2}A_{m+3} \parallel B_{m+3}A_{m+4} \parallel B_{m+4}A_{m+3} \parallel \cdots$$
$$\cdots \parallel B_{2m+2}A_{2m+3} \parallel A_1A_{2m+4}.$$

and

$$A_{m+2}A_{m+3} = B_{m+3}A_{m+4} = B_{m+4}A_{m+3} = \ldots$$
$$\cdots = B_{2m+2}A_{2m+3} = A_1A_{2m+4}.$$

(See Figure 9 for the case $m = 2$).

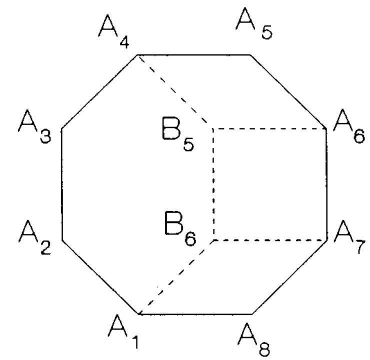


FIGURE 9

Now $A_1A_2 \ldots A_{m+2}B_{m+3}B_{m+4} \ldots B_{2m+2}$ forms a $(2m+2)$-sided parpolygon, and hence can be dissected into rhombi. Notice too that

$$A_{m+2}A_{m+3}A_{m+4}B_{m+3},$$
$$\vdots,$$
$$B_{2m+1}A_{2m+2}A_{2m+3}B_{2m+2},$$
$$B_{2m+2}A_{2m+3}A_{2m+4}A_1$$

are all rhombi, and hence the parpolygon with vertices

$$A_1, A_2, \ldots, A_{2m+4}$$

can be dissected into rhombi. Hence by PMI2, the result follows. $\square$

In all the variants of mathematical induction introduced so far, one of the induction steps involved assuming a statement is true and proving that the immediate successive statement could be proven true based on this assumption. In the next variant on the PMI, we modify this condition and have the truth of a statement in a sequence depending on the truth of all of its predecessors.

THE PRINCIPLE OF MATHEMATICAL INDUCTION, VARIANT 3 (PMI3). *Consider an infinite sequence of mathematical propositions*

$$P(1), P(2), P(3), \ldots$$

Suppose that

(1) *$P(1)$ is true (that is, can be proved),*

(2) *For each $m \in \mathbb{N}$, $[P(1)$ and $P(2)$ and ... and $P(m) \implies P(m+1)]$ (that is, we can prove $P(m+1)$ assuming the truth of $P(1)$, $P(2), \ldots, P(m-1)$ and $P(m)$).*

Then all the propositions of the sequence must be true, that is, $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $Q(1)$ be the statement $P(1)$, $Q(2)$ the statement "$P(1)$ and $P(2)$", $Q(3)$ the statement "$P(1)$, $P(2)$ and $P(3)$" and so on. A check shows that the sequence $Q(1), Q(2), Q(3), \ldots$ satisfies the conditions of the original PMI, and thus $Q(n)$ is true for all $n \in \mathbb{N}$, and hence $P(n)$ is true for all $n \in \mathbb{N}$, as for each $n \in \mathbb{N}$, $Q(n) \implies P(n)$. □

EXAMPLE 3.3. *Given any natural number n, define the sequence*

$$S_1, S_2, S_3, \ldots$$

as follows:

$$S_1 = n \text{ and, for } k > 1,\ S_k = \begin{cases} \dfrac{S_{k-1}}{2} & \text{if } S_{k-1} \text{ is even.} \\ S_{k-1} + 1 & \text{if } S_{k-1} \text{ is odd.} \end{cases}$$

Prove that for all $n \in \mathbb{N}$ with $S_1 = n$, there is a $k \in \mathbb{N}$ such that $S_k = 1$.

SOLUTION. Let $P(j)$ be the statement "For a sequence defined as above, if there is an $i \in \mathbb{N}$ with $S_i = j$, there is a $k \in \mathbb{N}$ such that $S_k = 1$". $P(1)$ is true since $S_1 = 1$. Now suppose there is an $m \in \mathbb{N}$ such that for all $r \leq m$ we have that $P(r)$ is true. Now set $S_1 = m+1$. If $m+1$ is even, $r = \dfrac{m+1}{2} \leq m$, and hence using the fact that $P(r)$ is true, and $S_2 = r$, we must have that $P(m+1)$ is true. If $m+1$ is odd, then as $m \geq 1$, $m+1 \geq 3$ ($m+1$ is odd), so we have that $r = \dfrac{m+2}{2} \leq m$. Hence since $S_3 = r$ and $P(r)$ is true, we have again that $P(m+1)$ is true. Hence by the PMI3, $P(n)$ is true for all $n \in \mathbb{N}$. □

Another variant of the PMI is the Principle of the Smallest Number, which though seemingly obvious is surprisingly useful. It is in fact an equivalent of the PMI, although we will only prove that it is a consequence of the PMI.

THE PRINCIPLE OF THE SMALLEST NUMBER (PSN). *Let S be a non-empty set of natural numbers. Then S has a smallest member, that is, there is an $m \in S$ such that for all $s \in S$, $m \leq s$.*

Proof. Suppose S is a set of natural numbers with no smallest member. We will show that S is necessarily empty. Let $P(n)$ be the statement "For all natural numbers $k \leq n$, $k \notin S$". Now $P(1)$ is true, for if $1 \in S$, 1 would be the smallest member as all natural numbers are greater than or equal to 1. Now suppose $P(m)$ is true for some $m \in \mathbb{N}$. Suppose $P(m+1)$ were false, that is $m+1 \in S$. Then, as S has no smallest member, there is a $k \in S$, such that $k < m+1$. But then $k \leq m$, contradicting $P(m)$. Hence $P(m+1)$ must be true. Hence by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$, implying that S is empty. □

One of the most common uses of the PSN is the so-called "method of infinite descent". Suppose we have a sequence of mathematical statements

$$P(1), P(2), P(3), \ldots$$

that we wish to prove true. If we are able to prove the following: "For every $k \in \mathbb{N}$ such that $P(k)$ is false, there is a $m \in \mathbb{N}$ such that $P(m)$ is false and $m < k$". Then we may conclude by the PSN that $P(n)$ is true for all $n \in \mathbb{N}$. For if not, then the set of all k such that $P(k)$ is false is a non-empty set of the natural numbers which, by the statement above, has no smallest element, contradicting the PSN. We illustrate this method with the following example.

EXAMPLE 3.4. *[USA MATHEMATICAL OLYMPIAD, 1978] An integer n is called* good *if we can write*

$$a_1 + a_2 + \cdots + a_k = n,$$

where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

Given the information that the integers 33 through to 73 are good, prove every integer greater than 32 is good.

SOLUTION. Call an integer *bad* if it is not good. What we need to show is that there are no bad integers greater than 32. Suppose for a contradiction, that the set S of all bad integers greater than 32 is not empty. By the PSN, it has a smallest member m, say. Note that by the given information, $m \geq 74$.

Now suppose m is even. Put $p = \dfrac{m-8}{2}$. Now $p \geq 33$, and since m is the smallest integer in S, and $p < m$, $p \notin S$, which means p is good. So choose $a_1, \ldots, a_k$ such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$$

and

$$a_1 + a_2 + \cdots + a_k = p.$$

Notice that

$$4 + 4 + 2a_1 + 2a_2 + \cdots + 2a_k = 8 + 2p = m$$

and

$$\begin{aligned}&\frac{1}{4} + \frac{1}{4} + \frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k}\\ =&\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right)\\ =&\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\\ =&1.\end{aligned}$$

But this shows that m is good, a contradiction.

Similarly, if m is odd, we can reach a contradiction by putting $p = \dfrac{m-9}{2}$, and using the fact that $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ and

$$3 + 6 + 2a_1 + 2a_2 + \cdots + 2a_k = 9 + 2p = m.$$

Thus both cases lead to a contradiction, so we must have that S is empty, which implies all numbers greater than 32 are good. □

The last variant of the PMI we introduce is the rather tricky method of "Backwards" Induction. Here the induction step involves using the truth of a statement in a sequence to prove its predecessor.

THE PRINCIPLE OF BACKWARDS MATHEMATICAL INDUCTION (PBMI). *Consider an infinite sequence of mathematical propositions*

$$P(1), P(2), P(3), \ldots$$

Suppose that

(1) *For some infinite subset S of $\mathbb{N}$, $P(s)$ is true for all $s \in S$.*
(2) *For each $m \in \mathbb{N}$, $P(m+1) \implies P(m)$ (that is, we can prove $P(m)$ on the assumption that $P(m+1)$ is true).*

Then all the propositions of the sequence must be true, that is, $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$, and without loss of generality assume that $x_1 < x_2 < x_3 < \ldots$. We can rearrange the sequence of statements as follows:

$$P(x_1), P(x_1 - 1), P(x_1 - 2), \ldots, P(1), P(x_2), P(x_2 - 1), \ldots$$
$$\ldots, P(x_1 + 1), P(x_3), \ldots$$

Now let $Q(1) = P(x_1)$, $Q(2) = P(x_1 - 1)$, $Q(3) = P(x_1 - 2)$, and in general let $Q(n)$ be the nth statement in the above sequence. A check shows that the sequence $Q(1), Q(2), Q(3), \ldots$ satisfies the conditions of the original PMI, and thus $Q(n)$ is true for all $n \in \mathbb{N}$, and hence $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

The PBMI is usually used as follows: One of the variants of the PMI is used to prove part (1) for some set S (popular choices are multiples or powers of a fixed number), and then part (2) is proved directly.

EXAMPLE 3.5. *Prove that*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$$

for all $n \in \mathbb{N}$, where $x_1, x_2, \ldots, x_n$ are all positive real numbers. This result is known as the arithmetic-geometric mean inequality.

SOLUTION. Let $P(n)$ be "$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$". We show first by the PMI that the result is true for all $P(s)$, where $s \in \{2, 4, 8, 16, \ldots\}$.

Firstly $P(2)$ is true, since

$$\begin{aligned}
& (x_1 - x_2)^2 \geq 0 \\
\Longrightarrow\ & (x_1)^2 + 2x_1x_2 + (x_2)^2 \geq 4x_1x_2 \\
\Longrightarrow\ & (x_1 + x_2)^2 \geq 4x_1x_2 \\
\Longrightarrow\ & \frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}.
\end{aligned}$$

Now suppose $P(2^m)$ is true for some $m \in \mathbb{N}$. Then

$$\begin{aligned}
& \frac{x_1 + x_2 + \ldots x_{2^{m+1}}}{2^{m+1}} \\
&= \frac{1}{2}\left(\frac{x_1 + x_2 + \cdots + x_{2^m}}{2^m} + \frac{x_{2^m+1} + x_{2^m+2} + \cdots + x_{2^{m+1}}}{2^m}\right) \\
&\geq \sqrt{\left(\frac{x_1 + x_2 + \cdots + x_{2^m}}{2^m}\right) \times \left(\frac{x_{2^m+1} + x_{2^m+2} + \cdots + x_{2^{m+1}}}{2^m}\right)} \\
&\qquad \text{since } P(2) \text{ is true} \\
&\geq \sqrt{(x_1 x_2 \ldots x_{2^m})^{\frac{1}{2^m}} \times (x_{2^m+1} x_{2^m+2} \ldots x_{2^{m+1}})^{\frac{1}{2^m}}} \\
&\qquad \text{since } P(2^m) \text{ is true} \\
&= (x_1 x_2 \ldots x_{2^{m+1}})^{\frac{1}{2^{m+1}}}.
\end{aligned}$$

Hence $P(2^{m+1})$ is true, and thus $P(s)$ is true for all $s \in \{2, 4, 8, 16, \ldots\}$.

Now suppose for some $m \in \mathbb{N}$, $P(m+1)$ is true. Given positive real numbers $x_1, x_2, \ldots, x_m$, put $x_{m+1} = \frac{x_1 + x_2 + \cdots + x_m}{m}$. Since x_{m+1} is the mean of $x_1, x_2, \ldots, x_m$, the mean of $x_1, x_2, \ldots, x_m$ is the same as the mean of $x_1, x_2, \ldots, x_{m+1}$, that is,

$$\frac{x_1 + x_2 + \cdots + x_m}{m} = \frac{x_1 + x_2 + \cdots + x_{m+1}}{m+1}.$$

Thus

$$\begin{aligned}
x_{m+1} &= \frac{x_1 + x_2 + \cdots + x_{m+1}}{m+1} \\
&\geq (x_1 x_2 \ldots x_{m+1})^{\frac{1}{m+1}} \quad \text{using } P(m+1) \\
&\geq (x_1 x_2 \ldots x_m)^{\frac{1}{m+1}} (x_{m+1})^{\frac{1}{m+1}}.
\end{aligned}$$

Thus

$$(x_{m+1})^{1-\frac{1}{m+1}} \geq (x_1 x_2 \ldots x_m)^{\frac{1}{m+1}},$$

that is,

$$(x_{m+1})^{\frac{m}{m+1}} \geq (x_1 x_2 \ldots x_m)^{\frac{1}{m+1}},$$

and hence

$$x_{m+1} = \frac{x_1 + x_2 + \cdots + x_m}{m} \geq (x_1 x_2 \ldots x_m)^{\frac{1}{m}}.$$

Thus $P(m)$ is true, and hence by PBMI the result follows. □

EXERCISES 3.

(1) Recall the Fibonacci sequence $F_1, F_2, F_3, \ldots$ introduced in exercises 2 (5). Prove that for $n, m \in \mathbb{N}$, $n > 1$ we have

$$F_{n+m} = F_{n-1}F_m + F_n F_{m+1}.$$

(2) [*Math. Digest, Jan. 1984*] Prove that for all $n \geq 4$, it is possible to dissect any non-equilateral triangle into n isosceles triangles.

(3) [**Mathematical Induction**, *Slinko*] On each planet in a planetary system consisting of an odd number of planets there is a astronomer observing the nearest planet. The distances between each pair of planets are all different. Prove that at least one planet is not observed by an astronomer.

(4) A sequence $a_1, a_2, a_3, \ldots$ is defined as follows: $a_1 = -2$, $a_2 = -16$ and $a_{n+2} = 8a_{n+1} - 15a_n$ for $n \in \mathbb{N}$. Show that $a_n = 3^n - 5^n$ for all $n \in \mathbb{N}$.

(5) [**Mathematical Induction**, *Slinko*] Suppose for a positive real number a, $a + \frac{1}{a}$ is an integer. Show that $a^n + \frac{1}{a^n}$ is an integer for all $n \in \mathbb{N}$.

(6) [*Tournament of Towns, 1979*] Consider all the possible subsets of the set $\{1, 2, \ldots, n\}$ which do not contain any consecutive numbers. For each of these subsets, the product of their elements is squared. Prove that the sum of all of these squared products is $(n+1)! - 1$.

(7) [**Mathematical Induction**, *Slinko*] Prove that an arbitrary sum of n cents can be paid by 3 cent and 5 cent coins, where n is an integer greater than 7.

(8) [**Mathematical Induction**, *Slinko*] Circular counters of equal diameter are lying on a table so that some of them touch each

other but do not overlap. Prove that these counters can be painted with four colours so that no counters of the same colour are touching.

(9) A real-valued function $f(x)$ is called *convex* if for any a, b real numbers, $f(\frac{a+b}{2}) \leq \frac{f(a)+f(b)}{2}$ (see Figure 10 for a graphical idea of the notion of a convex function). Prove that for any convex function $f(x)$ and any $n \in \mathbb{N}$, and real numbers $x_1, x_2, \ldots, x_n$,

$$f(\frac{x_1 + x_2 + \cdots + x_n}{n}) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

(This result is known as *Jensen's Inequality*.)

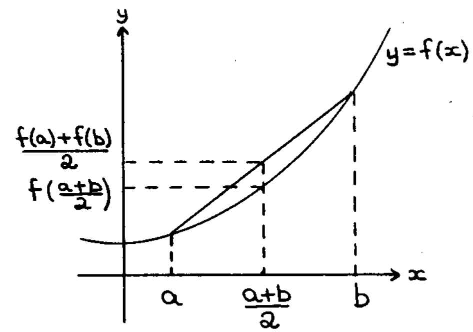


FIGURE 10

(10) Suppose I have an unlimited supply of coins of denominations a cents and b cents, where a and b are natural numbers with highest common factor 1. Show that I can always pay an amount of n cents, where n is any integer greater that $ab - a - b$.

(11) [*Moscow Mathematical Olympiad, 1979*] The area of the union of several circles (that is, the area of covered by all points lying inside at least one of the circles) equals 1. Prove that it is possible to choose several of them that do not intersect each other and whose total area is greater than $\frac{1}{9}$.

4. Harder Induction Problems

In this last section we will look at some rather difficult problems that can be solved using mathematical induction. Sometimes the choice of the induction parameter can be quite subtle, as the next example shows.

EXAMPLE 4.1. *A collection of natural numbers is said to have* property P *if the following holds: If any member of the collection is removed, the remaining numbers may be divided into two groups with equal sums. Prove that any collection of* $2n+1$ *natural numbers* $x_1, x_2, \ldots, x_{2n+1}$ *with property P, necessarily has all members equal to each other.*

SOLUTION. We use the PMI3 on the maximum value of

$$x_1, x_2, \ldots, x_{2n+1}.$$

If the maximum value of $x_1, x_2, \ldots, x_{2n+1}$ is 1, then all members of the set must be equal to 1, and the result holds. Suppose the result holds for all sets of $2n+1$ natural numbers whose maximum value is less than or equal to m. Let $x_1, x_2, \ldots, x_{2n+1}$ be a collection of natural numbers with maximum value $m+1$ that satisfies property P. Let $S = x_1 + x_2 + \cdots + x_{2n+1}$.

First note that the numbers in the collection are either all even or all odd: For, if we pick i, j such that $i, j \in \{1, 2, \ldots, 2n+1\}$, then there are natural numbers A_i and A_j such that $S = x_i + 2A_i = x_j + 2A_j$ by property P, that is x_i and x_j must have the same parity.

Suppose all the numbers in the collection are even. Then the collection

$$\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_{2n+1}}{2}$$

has property P and its maximum value is less than or equal to m. Hence by our assumption all the $\frac{x_i}{2}$'s are equal, and thus all x_i's are equal.

Suppose all the numbers in the collection are odd. Then the collection

$$\frac{x_1+1}{2}, \frac{x_2+1}{2}, \ldots, \frac{x_{2n+1}+1}{2}$$

has property P and its maximum value is less than or equal to m. Hence by our assumption all the $\frac{x_i+1}{2}$'s are equal, and hence all x_i's are equal.

Thus by the PMI3, the result holds. □

Another complication is that a problem may have two or more varying parameters. One approach is to first use the PMI on one of these parameters, and then apply the PMI to the other. Occasionally, though, the two parameters can be combined into one, as in the next problem.

EXAMPLE 4.2. *[***Mathematical Induction***, Slinko] Let* $x_1, \ldots, x_m$ *and* $y_1, \ldots, y_n$ *be natural numbers such that*

$$x_1 + \cdots + x_m = y_1 + \cdots + y_n < mn.$$

Prove that it is possible to cross out from the equation at least two terms (but not all of them) such that the equation remains true.

SOLUTION. We will apply the PMI1 to $s = m + n$. As

$$m \leq x_1 + \cdots + x_m < mn,$$

we conclude that $n > 1$ and similarly $m > 1$. Hence the minimum value of $s = 4$, when $m = n = 2$. For this case the result holds trivially as the only possible equations are $1+1 = 1+1$ and $1+2 = 1+2$. Now, suppose the result holds for some $s = k \in \mathbb{N}, k \geq 4$. Let

$$x_1 + \cdots + x_m = y_1 + \cdots + y_n < mn,$$

where $m + n = k + 1$. We may suppose, without loss of generality, that x_1 and y_1 are the largest numbers on the left-hand and right-hand sides respectively of the above equation. If $x_1 = y_1$, we are done as we can cross both these terms out. Hence, we may suppose without loss of generality that $x_1 > y_1$. Then consider the equation

$$(x_1 - y_1) + x_2 + \cdots + x_m = y_2 + \cdots + y_n.$$

In order to use our assumption for $s = k$, we need to show that $y_2 + \cdots + y_n < m(n-1)$: If $M = y_1 + \cdots + y_n$, then $y_1 \geq \frac{M}{n}$ and

$$y_2 + \cdots + y_n \leq M\frac{n-1}{n} < m(n-1).$$

Since $m+n-1 = k$, using the assumption above we can cancel at least two terms from the equation

$$(x_1 - y_1) + x_2 + \cdots + x_m = y_2 + \cdots + y_n.$$

(We regard $(x_1 - y_1)$ as one term.) Hence we can cancel out at least two terms from the original equation, and hence by the PMI1, our result holds. □

The last set of exercises contains some fairly hard problems of the sort that appear in international mathematics competitions. Most of them use a combination of the ideas that have been developed in this booklet, together with a few more clever tricks that require some innovative thought.

EXERCISES 4.

(1) [*Kürschak Mathematics Competition, 1932*] Prove that if a, b and n are positive integers and b is divisible by a^n, then $(a+1)^b - 1$ is divisible by a^{n+1}.

(2) [*All Union Mathematical Olympiad, 1971*] Prove that for any positive integer n there exists a positive integer which is divisible by 2^n and whose decimal representation consists only of digits 1 and 2.

(3) [*Chinese Mathematical Olympiad, 1979*] 2^n balls lie in several boxes. Consider the following procedure: select two boxes, and if the first one contains p balls and the other contains q balls $(p \geq q)$, to remove q balls from the first box and to put them in the the other box. Prove, that by repeating this procedure several times, it is possible to collect all the balls in a single box.

(4) Prove that for every rational number a and for all $n \in \mathbb{N}$, the equation $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = a$ has only finitely many positive integer solutions.

(5) [*All Union Mathematical Olympiad, 1966*] The numbers

$$a_1, a_2, \ldots, a_n$$

satisfy the conditions:

$$\begin{aligned} 0 \leq a_1 &\leq a_2 \leq 2a_1 \\ a_2 &\leq a_3 \leq 2a_2 \\ &\vdots \\ a_{n-1} &\leq a_n \leq 2a_{n-1}. \end{aligned}$$

Prove that it is possible to choose the signs in the sum

$$s = \pm a_1 \pm a_2 \pm \cdots \pm a_n$$

in such a way that the condition $0 \leq s \leq a_1$ is satisfied.

(6) [**Mathematical Induction**, *Slinko*] Let $x_1, x_2, \ldots, x_{2^n}$ be a sequence of non-negative integers. Transform this sequence into another sequence

$$|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_{2^n} - x_1|.$$

Prove that after several such transformations a sequence of only zeroes is obtained.

(7) Consider the following problem: Given a circle, choose any n distinct points on the circumference, and draw in all possible

chords with end-points any of these points. Let S_n be the maximum amount of regions into which such a procedure can divide a circle. Our aim will be to find a formula for S_n.

(a) Find S_2, S_3, S_4, S_5. Can you detect any pattern? Now try S_6 and see what happens!

(b) Show that for a set $\{1,2,3,\ldots,n\}$, it is possible to choose r different numbers from the set in $\dfrac{n!}{r!(n-r)!}$ ways (note $r \le n$, and by convention we let $0! = 1$). We will use the notation

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

(c) Suppose a circle has n chords, no 3 concurrent, with m internal intersection points. Show that the circle is divided into $n+m+1$ regions.

(d) Using all of the above, show that $S_n = 1 + \binom{n}{2} + \binom{n}{4}$.

(8) **[Polynomials,** *E.J. Barbeau*] Let x and y be natural numbers such that

$$|(y-x)y - x^2| = 1.$$

Show that $y-x, x$ and y are consecutive terms in the Fibonacci sequence introduced in exercises 2 (5).

5. Hints and Tips to the Exercises

Section 2.

(2) First show $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

(4) Use $a^{m+1} - b^{m+1} = a(a^m - b^m) + (a-b)b^m$.

(5) (b) For the induction step, show

$$F_{m+3}F_{m+1} - (F_{m+2})^2 = (F_{m+1})^2 - F_{m+2}F_m.$$

(c) Let $P(n)$ be the statement

$$F_{n+5} = 5F_{n+1} + 3F_n.$$

(6) Let $P(n)$ be the statement

$$\frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n} \le \frac{1}{\sqrt{2n+1}}.$$

(7) Let $P(n)$ be "$\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \le 2 - \dfrac{1}{n}$".

Section 3.

(1) Use the PMI3 on parameter m.

(2) For the base case, show first that every right-angled triangle can be divided into two isosceles triangles, and proceed using the PMI1.

(3) Use PMI2 with $a = 3$ and $d = 2$, with the number of planets as the induction parameter. Also, notice that for the two planets closest together, the astronomers observe each other.

(4) Use the PMI3 with parameter n.

(5) Let $P(n)$ be the statement: "There is an integer k_n such that $(a + \frac{1}{a})^n = a^n + \frac{1}{a^n} + k^n$" and use the PMI3.

(6) Use PMI3. In the induction step, when considering all possible subsets of $\{1,2,\ldots,m,m+1\}$, consider those sets that contain $m+1$ and those sets that do not contain $m+1$.

(8) Let $P(n)$ be the statement "Let x and y be distinct natural numbers such that

$$|(y-x)y-x^2| = 1,$$

with $y \leq F_{n+1}$. Then $y-x, x$ and y are consecutive terms in the Fibonacci sequence." Use exercise 2 no 5(b).

6. Answers to the Exercises

Section 1. The cases for the 3×3 grid can be solved as in Figure 11.

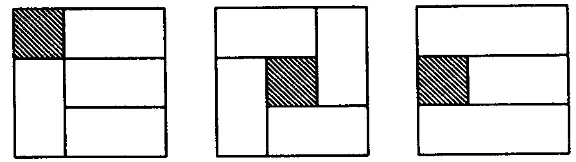


FIGURE 11

Suppose we can solve any $m \times m$ grid. Consider an $(m+1) \times (m+1)$ grid with one square shaded. We can always rotate the grid so that the bottom row and the rightmost column do not contain the shaded square. Now solve the problem for the $m \times m$ grid obtained by deleting the bottom row and rightmost column of the $(m+1) \times (m+1)$ grid. With that done, we can solve the $(m+1) \times (m+1)$ grid as in Figure 12 if m is even, and as in figure 13 if m is odd.

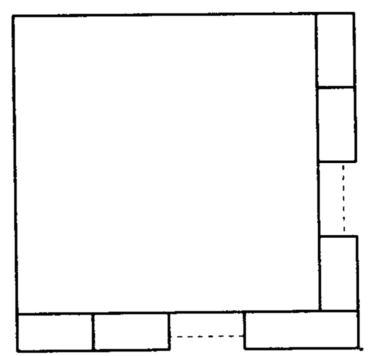


FIGURE 12

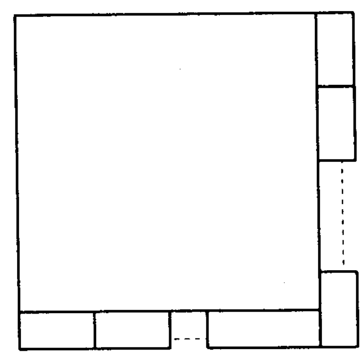


FIGURE 13

Section 2.

(1) For $n = 1$, $\frac{1}{1 \times 2} = \frac{1}{1+1}$ is true. Now suppose

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{m(m+1)} = \frac{m}{m+1}$$

for some $m \in \mathbb{N}$. Then

$$\begin{aligned} &\frac{1}{1 \times 2} + \cdots + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} \\ &= \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} \\ &= \frac{m^2 + 2m + 1}{(m+1)(m+2)} \\ &= \frac{m+1}{m+2}. \end{aligned}$$

Hence the result holds by the PMI.

(2) We first use the PMI to show $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$: For $n = 1$, we have $1 = \frac{1(1+1)}{2}$ is true. Now suppose

$$1 + 2 + 3 + \cdots + m = \frac{m(m+1)}{2}$$

for some $m \in \mathbb{N}$. Then

$$1 + 2 + 3 + \cdots + m + (m+1) = \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2},$$

so the result follows by the PMI.

We now use the PMI again to show

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \in \mathbb{N}$: For $n = 1$, $1^3 = \frac{1^2(1+1)^2}{4}$ is true. Now suppose $1^3 + 2^3 + 3^3 + \cdots + m^3 = \frac{m^2(m+1)^2}{4}$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned} &1^3 + 2^3 + 3^3 + \cdots + m^3 + (m+1)^3 \\ &= \frac{m^2(m+1)^2}{4} + (m+1)^3 \\ &= \frac{(m+1)^2(m^2 + 4m + 4)}{4} \\ &= \frac{(m+1)^2(m+2)^2}{4}. \end{aligned}$$

Thus the result follows by the PMI.

Combining the two results, we have, for all $n \in \mathbb{N}$,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = (1 + 2 + \cdots + n)^2.$$

(3) For $n = 1$, $2^{1+4} = 32 \geq (1+4)^2 = 25$ is true. Now suppose $2^{m+4} \geq (m+4)^2$ for some $m \in \mathbb{N}$. We have

$$\begin{aligned} 2^{m+5} &= 2(2^{m+4}) \\ &\geq 2(m+4)^2 \\ &= 2m^2 + 16m + 32 \\ &> m^2 + 10m + 25 \\ &= (m+5)^2. \end{aligned}$$

Thus by the PMI, the result follows.

(4) For $n = 1$, $a^1 - b^1$ is obviously divisible by $a - b$. Now suppose $a^m - b^m$ is divisible by $a - b$. Hence $a(a^m - b^m)$ is divisible by $a - b$. Also $(a - b)b^m$ is clearly divisible by $a - b$, hence $a(a^m - b^m) + (a - b)b^m = a^{m+1} - b^{m+1}$ is divisible by $a - b$ and the result holds by the PMI.

(5) (a) For $n = 1$, $F_1 = 1$ and $F_3 = 2$ imply $F_1 = F_3 - 1$. Now suppose $F_1 + F_2 + \cdots + F_m = F_{m+2} - 1$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned} & F_1 + F_2 + \cdots + F_m + F_{m+1} \\ &= F_{m+2} - 1 + F_{m+1} \\ &= F_{m+3} - 1 \\ &\qquad \text{since } F_{m+3} = F_{m+1} + F_{m+2}. \end{aligned}$$

Thus by the PMI, the result follows.

(b) For $n = 1$, we have $|F_3F_1 - (F_2)^2| = |2-1| = 1$ is true. Now suppose, for some $m \in \mathbb{N}$, $|F_{m+2}F_m - (F_{m+1})^2| = 1$. Then

$$\begin{aligned} & |F_{m+3}F_{m+1} - (F_{m+2})^2| \\ &= |(F_{m+2} + F_{m+1})F_{m+1} - (F_{m+2})^2| \\ &\qquad \text{since } F_{m+3} = F_{m+2} + F_{m+1} \\ &= |F_{m+2}(F_{m+1} - F_{m+2}) + (F_{m+1})^2| \\ &= |F_{m+2}(F_{m+1} - (F_{m+1} + F_m)) + (F_{m+1})^2| \\ &\qquad \text{since } F_{m+2} = F_{m+1} + F_m \\ &= |-F_{m+2}F_m + (F_{m+1})^2| \\ &= |F_{m+2}F_m - (F_{m+1})^2| \\ &= 1. \end{aligned}$$

Hence by the PMI, the result follows.

(c) Let $P(n)$ be the statement "$F_{n+5} = 5F_{n+1} + 3F_n$". $P(1)$ is true since $F_6 = 8 = 5F_2 + 3F_1$. Now suppose $P(m)$ is true for some $m \in \mathbb{N}$, $F_{m+5} = 5F_{m+1} + 3F_m$. Now, using repeated applications of the rule $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$, we have

$$\begin{aligned} F_{m+4} &= F_{m+3} + F_{m+2} \\ &= F_{m+2} + F_{m+1} + F_{m+2} \\ &= F_{m+1} + 2F_{m+2} \\ &= F_{m+1} + 2(F_{m+1} + F_m) \\ &= 3F_{m+1} + 2F_m. \end{aligned}$$

Using this, the induction hypothesis and another application of the rule above, we have

$$\begin{aligned} F_{(m+1)+5} &= F_{m+6} \\ &= F_{m+5} + F_{m+4} \\ &= 5F_{m+1} + 3F_m + 3F_{m+1} + 2F_m \\ &= 3F_{m+1} + 5(F_{m+1} + F_m) \\ &= 3F_{m+1} + 5F_{m+2}. \end{aligned}$$

Thus $P(m+1)$ is true. Hence by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$.

To see how this helps to answer the question, notice that for each $n \in \mathbb{N}$, F_n and F_{n+5} have the same remainder when divided by 5. By looking at the first 5 Fibonnaci numbers $1, 1, 2, 3, 5$, only the fifth number is divisible by 5. Hence, using an inductive argument again, we can conclude that n is divisible by 5 if and only if F_n is divisible by 5.

(6) Let $P(n)$ be the statement

$$\frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n} \leq \frac{1}{\sqrt{2n+1}}.$$

It is a fairly trivial observation that $P(1)$ is true. Now suppose for some $m \in \mathbb{N}$, $P(m)$ is true. Then

$$\begin{aligned}
&\frac{1\times 3\times 5\times\cdots\times(2m-1)\times 2m+1}{2\times 4\times 6\times\cdots\times 2m\times 2(m+1)}\\
&\le \frac{1}{\sqrt{2m+1}}\times\frac{2m+1}{2(m+1)}\\
&= \frac{\sqrt{2m+1}}{2m+2}\\
&= \sqrt{\frac{2m+1}{4m^2+8m+4}}\\
&\le \sqrt{\frac{2m+1}{4m^2+8m+3}}\\
&= \sqrt{\frac{2m+1}{(2m+1)(2m+3)}}\\
&= \frac{1}{\sqrt{2m+3}}\\
&= \frac{1}{\sqrt{2[m+1]+1}}.
\end{aligned}$$

Hence $P(m+1)$ is true, and by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$. To complete the problem, one need only note that

$$\frac{1}{\sqrt{2n+1}} \le \frac{1}{\sqrt{2n}}$$

for all $n \in \mathbb{N}$, so the original statement is true for all $n \in \mathbb{N}$.

(7) Let $P(n)$ be "$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \le 2-\frac{1}{n}$". $P(1)$ is true as $\frac{1}{1^2} \le 2-\frac{1}{1}$. Now suppose $P(m)$ is true for some $m \in \mathbb{N}$, that is,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2} < 2-\frac{1}{m}.$$

To see that this implies $P(m+1)$ is true, notice that

$$\begin{aligned}
\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2}+\frac{1}{(m+1)^2} &\le 2-\frac{1}{m}+\frac{1}{(m+1)^2}\\
&\qquad\text{since } P(m) \text{ is true}\\
&= 2-\frac{m^2+m+1}{m(m+1)^2}\\
&< 2-\frac{m^2+m}{m(m+1)^2}\\
&= 2-\frac{1}{m+1}.
\end{aligned}$$

Thus by the PMI, $P(n)$ is true for all $n \in \mathbb{N}$, and the result follows since since $2-\frac{1}{n} < 2$ for all $n \in \mathbb{N}$.

Section 3.

(1) We use the PMI3 on the parameter m. For $m = 1$, $F_{n+1} = F_{n-1} + F_n = F_{n-1}F_1 + F_nF_2$ is true. For $m = 2$, $F_{n+2} = F_{n+1} + F_n = F_n + F_{n-1} + F_n = F_{n-1}F_2 + F_nF_3$. Now suppose the result is true for all $m \le k$, for some $k \in \mathbb{N}$ with $k > 1$. Then $F_{n+k+1} = F_{n+k} + F_{n+k-1} = F_{n-1}F_k + F_nF_{k+1} + F_{n-1}F_{k-1} + F_nF_k = F_{n-1}(F_k + F_{k-1}) + F_n(F_{k+1} + F_k) = F_{n-1}F_{k+1} + F_nF_{k+2}$. Hence the result is true for $m = k+1$, and hence by the PMI3, the result follows.

(2) We use PMI1 with $k = 4$ and parameter the number of isosceles triangles. It is always possible to dissect a right-angled triangle into two isosceles triangles, using the dissection given in Figure 14.

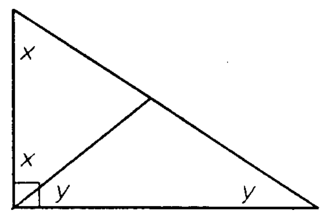


FIGURE 14

Hence, for any non- equilateral triangle, if we drop a perpendicular from the vertex opposite the longest side, we can dissect each of the two resulting right-angled triangles into two isosceles triangles, yielding a dissection of the triangle into four isosceles triangles (see Figure 15).

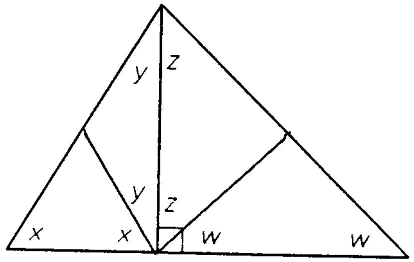


FIGURE 15

Now suppose that we can dissect any non-equilateral triangle into m isosceles triangles. Now, given any triangle ABC with shortest side AB say, label the point D on BC such that $BD = AB$ (see Figure 16). Now $\angle C$ is the largest angle as it is opposite the shortest side AB, and hence it cannot be $60°$ as ABC is not equilateral. Thus triangle ADC is non-equilateral. We apply the induction hypothesis and dissect ADC into m isosceles triangles, thereby dissecting ABC into $m + 1$ isosceles triangles. Hence by the PMI1, the result follows.

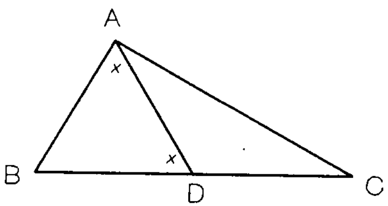


FIGURE 16

(3) We use PMI2 with $a = 3$, $d = 2$ on the number of planets. For three planets, the two observers on the two planets closest

to each other observe each other's planets, and hence the third planet remains unobserved. Now suppose, for some $m \in \mathbb{N}$, the result is true for $2m + 1$ planets. Consider a $2m + 3$ planetary system. Again, pick the two planets closest to each other, and notice that the two astronomers observe each other's planets. This leaves $2m + 1$ planets. If an astronomer on any of these planets is observing one of the two planets mentioned before, then one of the other $2m + 1$ planets will be unobserved. If nobody does this, then we can consider the $2m + 1$ planets as a planetary system all on its own, and use our assumption to conclude there must be a unobserved planet among them. Hence the result is true for $2m + 3$ planets. Hence, by the PMI2, the result holds.

(4) We use PMI3. The result is clearly true for $n = 1$ and 2. Suppose for some $k > 1$, $a_m = 3^m + 5^m$ for all $m \leq k$. Then,

$$\begin{aligned} a_{m+1} &= 8a_m - 15a_{m-1} \\ &= 8(3^m - 5^m) - 15(3^{m-1} + 5^{m-1}) \\ &= 3^{m-1}(8 \times 3 - 15) - 5^{m-1}(8 \times 5 - 15) \\ &= 3^{m+1} - 5^{m+1}. \end{aligned}$$

Hence, by the PMI3, the result is true.

(5) Let $a + \frac{1}{a}$ be an integer. Let $P(n)$ be the statement "There is an integer k_n such that $(a+\frac{1}{a})^n = a^n + \frac{1}{a^n} + k_n$". $P(1)$ is trivially true, and $P(2)$ is true as

$$(a+\frac{1}{a})^2 = a^2 + \frac{1}{a^2} + 2.$$

Now suppose, for some $m > 1$, $P(k)$ is true for all $k \le m$. Then, since $P(m)$ and $P(m-1)$ are true,

$$\begin{aligned}
&(a+\frac{1}{a})^{m+1} \\
&= (a+\frac{1}{a})^m (a+\frac{1}{a}) \\
&= (a^m + \frac{1}{a_m} + k_m)(a+\frac{1}{a}) \\
&= a^{m+1} + \frac{1}{a^{m+1}} + a^{m-1} + \frac{1}{a^{m-1}} + k_m(a+\frac{1}{a}) \\
&= a^{m+1} + \frac{1}{a^{m+1}} + (a+\frac{1}{a})^{m-1} - k_{m-1} + k_m(a+\frac{1}{a}).
\end{aligned}$$

Now, since $a+\frac{1}{a}$ is an integer,

$$k_{m+1} = (a+\frac{1}{a})^{m-1} - k_{m-1} + k_m(a+\frac{1}{a})$$

is an integer, and hence $P(m+1)$ is true. Hence by the PMI3, the result follows.

(6) Let S_n be the sum of the squares of the products of the numbers in all possible subsets of the set $\{1, 2, \ldots, n\}$ that do not contain any consecutive numbers. Let $P(n)$ be the statement "$S_n = (n+1)! - 1$". We use PMI3. $P(1)$ is true since $S_1 = 1^2 = 2! - 1$. $P(2)$ is true, for the only valid subsets are $\{1\}$ and $\{2\}$, and $S_2 = 1^2 + 2^2 = 3! - 1$.

Now suppose for some $m \in \mathbb{N}, m > 1$, we have that $P(m)$ and $P(m-1)$ are true. We derive the formula for S_{m+1} as the

sum of two parts, namely, those sets that contain $m+1$, and those sets that do not. Firstly, the subsets containing $m+1$ and at least one other member, since it cannot contain m, will contribute $(m+1)^2 S_{m-1}$ to the sum. Secondly, subsets that do not contain $m+1$ will contribute S_m to the sum. Lastly, the singleton set $\{m+1\}$ contributes $(m+1)^2$ to the sum. Hence

$$\begin{aligned}
S_{m+1} &= S_m + (m+1)^2 S_{m-1} + (m+1)^2 \\
&= (m+1)! - 1 + (m+1)^2(m! - 1 + 1) \\
&= (m+1)! - 1 + (m+1)((m+1)!) \\
&= (m+2)(m+1)! - 1 \\
&= (m+2)! - 1.
\end{aligned}$$

Thus $P(m+1)$ is true, and by the PMI3, the result follows.

(7) Let S be a subset of natural numbers defined as follows: $x \in S$ if and only if $x \ge 7$ and x cents cannot be paid out out using only 3 and 5 cent coins. We need to show that S is empty. Suppose not, then by the PSN, S has a smallest member, say m. Now since $8 = 3+5, 9 = 3+3+3, 10 = 5+5$, 8, 9 and 10 are not in S, hence $m - 3 > 7$. Now, suppose we can pay out $m-3$ cents. Then, adding a 2 cent coin, we would be able to pay out m cents, a contradiction. Hence $m - 3 \in S$, contradicting m is the smallest member. Hence S must be empty, and the result follows.

(8) We use the PSN. Suppose there is an arrangement of counters that requires more than 4 colours. Let S be the set of all natural numbers n such that there is an arrangement of n counters that requires more than 4 colours. By the PSN, S has a smallest member, say m. Notice that $m > 4$. Now consider an arrangement of m counters requiring more than 4 colours. Consider the centre points of the m counters, and all possible straight lines joining two centre points. A subset of these straight lines will form a polygon which contains all the other lines. At least one of the vertices of this polygon must have angle less than 180°. Call this vertex O. Now if the counter with centre O is touched by two other counters with centres P and Q (see Figure 17), then as $PQ \ge PO$ and $PQ \ge QO$, $\angle POQ \ge 60°$, and thus the

counter with centre O can only be touched by at most 2 other counters.

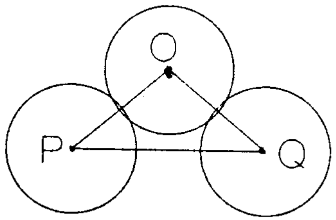


FIGURE 17

If we remove this counter, by the mininimality of m, we can colour the arrangement using only four colours. But then we can colour the original arrangement using only four colours, by colouring the counter at O with a colour not used by any counter touching it. But this contradicts $m \in S$, hence S must be empty and the result follows.

(9) We use the PBMI. For $n = 1$, the result is trivially true, and for $n = 2$ the result follows from the convexity of f.

We next show that if the statement is true for some $n = m$, then it is true for all $n = 2m$, and hence the statement is true for all powers of 2. Suppose the result is true for some $n = m$, and let $x_1, x_2, \ldots, x_{2m}$ be real numbers. Now

$$\begin{aligned}
& f\left(\frac{x_1 + x_2 + \cdots + x_{2m}}{2m}\right) \\
\leq\, & \frac{1}{2}\left(f\left(\frac{x_1 + \cdots + x_m}{m}\right) + f\left(\frac{x_{m+1} + \cdots + x_{2m}}{m}\right)\right) \\
& \quad \text{using the convexity of } f \\
\leq\, & \frac{1}{2}\left(\frac{f(x_1) + \cdots + f(x_m)}{m} + \frac{f(x_{m+1}) + \cdots + f(x_{2m})}{m}\right) \\
& \quad \text{using the case } n = m \\
=\, & \frac{f(x_1) + f(x_2) + \cdots + f(x_{2m})}{2m}.
\end{aligned}$$

Hence the result holds for $n = 2m$.

Now let $m \in \mathbb{N}$, and suppose the result holds for $n = m + 1$. We show the result holds for $n = m$. Let $x_1, x_2, \ldots, x_m$ be any real numbers. Put $x_{m+1} = \dfrac{x_1 + x_2 + \cdots + x_m}{m}$. Now

$$\begin{aligned}
& f\left(\frac{x_1 + x_2 + \cdots + x_m}{m}\right) \\
=\, & f\left(\frac{x_1 + x_2 + \cdots + x_{m+1}}{m+1}\right) \\
\leq\, & \frac{f(x_1) + \cdots + f(x_m) + f\left(\frac{x_1 + \cdots + x_m}{m}\right)}{m+1},
\end{aligned}$$

using the $n = m + 1$ case, from which

$$f\left(\frac{x_1 + x_2 + \cdots + x_m}{m}\right) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_m)}{m}.$$

Hence, by the PBMI, the result holds.

(10) We use PBMI. Let $P(n)$ be the statement "We can pay n cents using only coins of denominations a and b cents". We wish to show that $P(n)$ is true for all $n > ab - a - b$. Firstly note that $P(s)$ is true for all $s \in \{a, 2a, 3a, \ldots\}$. Now suppose, for some $m > ab - a - b$, we have that $P(m + 1)$ is true, that is, there are non-negative integers x, y such that $ax + by = m + 1$. Now $m+1 \geq ab-a-b+2$, and hence $ax+by-1 \geq ab-a-b+1$. Since the highest common factor of a and b is 1, there are integers p and q such that $pa + qb = 1$. (This is a standard result in number theory - see *Topics in Number Theory* by Valentin Goranko, which is Book number 2 in this series.) Hence $ax + by - pa - qb + a + b \geq ab + 1$, which in turn gives

$$\frac{x - p + 1}{b} - \frac{q - y - 1}{a} \geq 1 + \frac{1}{ab} > 1.$$

Hence, there must be an integer t such that

$$\frac{x - p + 1}{b} > t > \frac{q - y - 1}{a}.$$

Now $q - y - 1 < at$ implies $at - q + y \geq 0$, and $x - p + 1 > bt$ implies $x - bt - p \geq 0$. Thus

$$\begin{aligned}&a(x - bt - p) + b(at - q + y)\\ &= ax + by - (ap + bq)\\ &= (m + 1) - 1\\ &= m.\end{aligned}$$

Thus $P(m)$ is true. Hence, by PBMI, $P(n)$ is true for all $n > ab - a - b$.

(11) Let $P(n)$ be the statement "If the union of n circles in the plane has area A, it is possible to choose from among them several pairwise non-intersecting circles, the sum of whose areas is greater than or equal to $\frac{A}{9}$". We use PMI3.

$P(1)$ is trivially true.

Now suppose for some $m \in \mathbb{N}$, $P(k)$ was true for all $k \leq m$. Let S be a set of $m+1$ circles with area of their union A. Let C be the circle of largest radius in S, and denote its radius by r and its area by A_1. If $A_1 \geq \frac{A}{9}$, then C has the required property, hence we may assume that $A_1 < \frac{A}{9}$.

Consider a circle C_1 of radius $3r$, concentric with C. The radius of each circle in S is no greater than r, and thus if such a circle intersects C, it lies completely in C_1. (See Figure 18.) Let S_1 be the set of all circles that intersect C (including C). The area of their union must be less than $9A_1 < A$. But then there is a circle in S that is not in S_1, and hence the set $S_2 = S \setminus S_1$ is not empty, and A_2 is the area of the union of circles in S_2, we have $A_2 \geq A - 9A_1$.

Now suppose S_2 contains k circles. Since $C \notin S_2$, $k \leq m$, and since $P(k)$ is assumed true, we may suppose there is a subset S_3 of S_2 consisting of pairwise non-intersecting circles

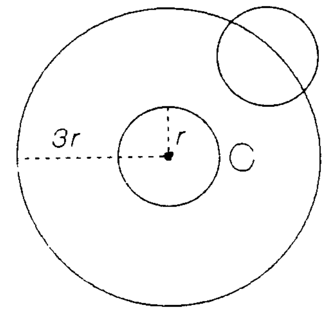


FIGURE 18

with area of their union A_3, and $A_3 \geq \frac{A_2}{9}$. Now $S_3 \bigcup \{C\}$ possesses the required properties to make $P(m + 1)$ true, as C does not intersect any of the circles in S_3, and the combined union of their areas is greater than or equal to

$$\frac{A_2}{9} + A_1 \geq \frac{A - 9A_1}{9} + A_1 = \frac{A}{9}.$$

Hence, by the PMI3, the result holds.

Section 4.

(1) We use the PMI on n, starting from $n = 0$. For $n = 0$ it is true, since $(a + 1)^b - 1$ is divisible by a. Now suppose that the proposition is true for some $n = k$, that is, for all positive integers a and b such that b is divisible by a^k, we have $(a+1)^b - 1$ is divisible by a^{k+1}. We will show that if a and b_1 are positive integers with b_1 divisible by a^{k+1}, then $(a+1)^{b_1} - 1$ is divisible by a^{k+2}.

Let b_1 be divisible by a^{k+1}. Let $b = \frac{b_1}{a}$. Then, since b is divisible by a^k, and thus by our assumption, $(a + 1)^b - 1$ is divisible by

a^{k+1}. Observe that

$$\begin{aligned}&(a+1)^{b_1}-1\\ &= (a+1)^{ab}-1\\ &= [(a+1)^b-1][(a+1)^{(a-1)b}+(a+1)^{(a-2)b}+\cdots+1].\end{aligned}$$

It thus remains to show that the second factor in the last expression is divisible by a. This is true as we can rewrite this factor as

$$[(a+1)^{(a-1)b}-1]+[(a+1)^{(a-2)b}-1]+\dots$$
$$\cdots+[(a+1)^b-1]+a.$$

This proves the required result.

(2) We prove a slightly stronger result: For each $n \in \mathbb{N}$, there is an n-digit number divisible by 2^n whose decimal representation consists only of the digits 1 and 2. We proceed by induction on n.

For $n = 1$, the number 2 satisfies the above properties. Now suppose the result is true for some $n = k$, that is, there is an k-digit number, say x, such that $x = 2^k \times y$ for some $y \in \mathbb{N}$, and the decimal representation of x consists only of 1's and 2's. Consider the two $(k+10)$-digit numbers $z_1 = 10^k + x$ and $z_2 = 2 \times 10^k + x$. Notice that their decimal representations consist of 1's and 2's only. We will prove one of them is divisible by 2^{k+1}. To show this, notice first that since 5^k is odd and 2×5^k is even, one of the integers $5^k + y$ and $2 \times 5^k + y$ is even. Hence one of $2^k(5^k + y) = z_1$ and $2^k(2 \times 5^k + y) = z_2$ is divisible by 2^{k+1}, and this completes the proof.

(3) We prove the result using the PMI on n. For $n = 1$ the stated proposition is trivially true. Now suppose that the statement is true for some $n = k$, and consider an arbitrary placement of 2^{k+1} balls in the boxes. Colour each box containing an odd number of balls red, and each box containing an even number of balls blue. Notice that since the total number of balls is even, and the total number of balls in blue boxes is even, thus the

total number of balls in red boxes is also even. Hence there must be an even amount of red boxes. Pair these boxes arbitrarily, and perform one move on each pair of boxes. In this way we obtain a placement of balls such that each box contains an even number of balls. It is possible now to group the balls in each box in turn in pairs, call each such pair a *two-ball*. We thus obtain 2^k two-balls, and notice that if applying moves for single balls has the same effect as applying the move for two-balls. We thus can use our assumption to solve the problem for the two-balls, and hence by the PMI the result holds.

(4) We proceed by induction on n. The result is clearly true if $n = 1$. Suppose it is true for $n = k$. Consider the equation

$$\frac{1}{x_n}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}} = a.$$

We may suppose without loss of generality that

$$x_1 \geq x_2 \geq \cdots \geq x_{k+1}.$$

Then

$$a = \frac{1}{x_k}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}+\frac{1}{x_{k+1}} \leq \frac{k+1}{x_{k+1}},$$

and hence $x_{k+1} \leq \dfrac{k+1}{a}$. Since $x_{k+1} \in \mathbb{N}$, there are only finitely many possible values for x_{k+1}. But then the equation

$$\frac{1}{x_n}+\frac{1}{x_2}+\cdots+\frac{1}{x_k} = a - \frac{1}{x_{k+1}}$$

has only finitely many solutions $x_1, \dots, x_2$, for each possible value of x_{n+1}, using our assumption. Hence the result follows.

(5) We proceed by induction on n. For $n = 1$, the result is trivially true. Suppose that the result holds for some $n = k$. Let $b_1, b_2, \dots, b_{k+1}$ be any numbers satisfying the inequalities

$$\begin{aligned}0 \leq b_1 \leq b_2 &\leq 2b_1\\ b_2 \leq b_3 &\leq 2b_2\\ \vdots \quad \vdots \quad &\vdots\\ b_k \leq b_{k+1} &\leq 2b_k.\end{aligned}$$

We need to show that the signs in the sum $s_1 = \pm b_1 \pm \cdots \pm b_{k+1}$ can be chosen in such a way such that $0 \le s_1 \le b_1$. Letting

$$\begin{aligned} a'_1 &= b_2, \\ a'_2 &= b_3, \\ &\vdots \\ a'_{k-1} &= b_k, \\ a'_k &= b_{k+1}, \\ s' &= \pm b_2 \pm \cdots \pm b_{k+1} \\ &= \pm a'_1 \pm \ldots a'_k, \end{aligned}$$

we can apply our assumption, deducing that the signs in s' can be chosen in such a way that $0 \le s' \le a'_1 = b_2$.

Compare s' and b_1. There are two cases:

$s' \le b_1$: Then $s_1 = b_1 - s'$ has the required property, for

$$0 \le s_1 = b_1 - s' \le b_1.$$

$s' > b_1$: Then $s_1 = -b_1 + s'$ has the required property, for

$$0 \le s_1 = s' - b_1 \le b_2 - b_1 \le 2b_1 - b_1 = b_1.$$

Thus $0 \le s_1 \le b_1$ in both cases, and hence the result is true for $n = k+1$ as well. Hence by the PMI, the result follows.

(6) We start by introducing two bits of notation. Given non-negative integers a and b, we denote by $a*b$ the number $|a-b|$, and define the *parity function*

$$p(a) = \begin{cases} 1 \text{ if } a \text{ is odd} \\ 0 \text{ if } a \text{ is even.} \end{cases}$$

The parity function and the operation $*$ obey the following rules for all non-negative integers a, b and c:

$$\begin{aligned} p((a*b)*c) &= p(a*(b*c)) \\ p(a*b) &= p(b*a) \\ p(a*b) &= p(a)*p(b) \\ p(a*a) &= 0 \\ p(p(a)) &= p(a). \end{aligned}$$

Before we consider the actual problem, consider an infinite sequence of non-negative integers

$$x_1, x_2, x_3, \ldots, x_m, \ldots$$

We can transform this sequence into another infinite sequence

$$x_1 * x_2, x_2 * x_3, x_3 * x_4, \ldots, x_m * x_{m+1}, \ldots$$

We make the following claim: For any $n \in \mathbb{N}$, after 2^n such transformations of the sequence

$$x_1, x_2, x_3, \ldots, x_m, \ldots$$

we get a sequence

$$y_1, y_2, y_3, \ldots, y_m, \ldots$$

such that

$$\begin{aligned} p(y_1) &= p(x_1 * x_{2^n+1}) \\ p(y_2) &= p(x_1 * x_{2^n+2}) \\ &\vdots \\ p(y_m) &= p(x_1 * x_{2^n+m}) \\ &\vdots \end{aligned}$$

We prove this claim by induction on n.

For $n = 1$, after 2 transformations we see, for each $m \in \mathbb{N}$, $y_m = (x_m * x_{m+1}) * (x_{m+1} * x_{m+2})$, and hence, applying the laws above we have $p(y_m) = p(x_m * x_{2+m})$.

Now suppose the result is true for some $n = k$. Consider an infinite sequence

$$x_1, x_2, x_3, \ldots, x_m, \ldots$$

and suppose after 2^{k+1} transformations it yields the sequence

$$z_1, z_2, z_3, \ldots, z_m, \ldots$$

We can divide the process into two parts: First apply 2^k transformations to the sequence

$$x_1, x_2, x_3, \ldots, x_m, \ldots$$

to get a sequence, say,

$$y_1, y_2, y_3, \ldots, y_m, \ldots$$

and then apply another 2^k transformations to this sequence to get the final sequence

$$z_1, z_2, z_3, \ldots, z_m, \ldots$$

Using our assumption that the result is true for $n = k$ twice, and applying the rules above, we have for any $m \in \mathbb{N}$,

$$\begin{aligned} p(z_m) &= p(y_m * y_{2^k+m}) \\ &= p(y_m) * p(y_{2^k+m}) \\ &= [p(x_m) * p(x_{2^k+m})] * [p(x_{2^k+m}) * p(x_{2^k+2^k+m})] \\ &= p(x_m * x_{2^k+2^k+m}) \\ &= p(x_m * x_{2^{k+1}+m}), \end{aligned}$$

so the result is true for $n = k + 1$, and hence by the PMI the claim is true.

Now to return to the original problem. Suppose I have a sequence of 2^n non-negative integers

$$x_1, x_2, \ldots, x_{2^n},$$

and apply the transformation stated in the problem 2^n times, by the claim above and the wrap-around effect of the transformation, we will obtain a sequence

$$y_1, y_2, \ldots, y_{2^n}$$

such that for $m = 1, \ldots, n$, $p(y_m) = p(x_m) * p(x_m) = 0$, that is a sequence with each member even.

We finish off the problem by applying the PMI3, but this time on the largest number in the sequence. Suppose we have a sequence

$$x_1, x_2, \ldots, x_{2^n},$$

and the largest number in the sequence is 1. Then the sequence consists only of 0's and 1's, and hence after 2^n transformations the sequence can consist only of even numbers, and it is not hard to see that these must all be 0's. Now suppose the result is true for all sequences with largest number in the sequence at most k, and consider a sequence

$$x_1, x_2, \ldots, x_{2^n}$$

whose largest member is is $k + 1$. After 2^n transformations we will be left with a sequence

$$y_1, y_2, \ldots, y_{2^n}$$

whose members are all even, and less than or equal to $k + 1$. Now the sequence

$$\frac{y_1}{2}, \frac{y_2}{2}, \ldots, \frac{y_{2^n}}{2}$$

has largest member less than $k+1$, and hence by our assumption after a number of transformations will reach a sequence of 0's. Hence the sequence $y_1, y_2, \ldots, y_{2^n}$ will reach a sequence of 0's, and thus the original sequence will reach a sequence of 0's. Hence the result is true for all sequences with largest number $k+1$, and thus by the PMI3, the result is true.

(7) (a) $S_2 = 2,\ S_3 = 4,\ S_4 = 8,\ S_5 = 16$ but $S_6 = 31$.

(b) Notice first that there is only one way of choosing 0 items from a set, and the formula reflects this. So we need only prove the formula for values of r greater than 0. To do this, we use the PMI on n. For the set $\{1\}$, it is possible to choose 1 item from this set in only one way, and $\binom{1}{1} = 1$, hence the formula is true for $n = 1$.

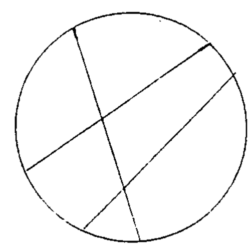
Now suppose the formula is true for some $n = k$. Consider the amount of ways of picking r different numbers from the set $\{1, 2, \ldots, k, k+1\}$. There are two possible cases: either your choice of r numbers contains $k+1$ or it does not. The number of ways of choosing r numbers which contain $k+1$ is the same as choosing $r-1$ numbers from the set $\{1, 2, \ldots, k\}$, and the number of ways of choosing r numbers not containing $k+1$ is equal to the number of ways of choosing r numbers from the set $\{1, 2, \ldots, k\}$. By our assumption, these numbers are equal to $\binom{k}{r-1}$ and $\binom{k}{r}$ respectively, and hence the number of ways of choosing r numbers from the set $\{1, 2, \ldots, k, k+1\}$ is

$$\begin{aligned}
\binom{k}{r-1} + \binom{k}{r} &= \frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{r!(k-r)!} \\
&= \frac{k! \times r + k! \times (k-r+1)}{r!(k-r+1)!} \\
&= \frac{k! \times (k+1)}{r!((k+1)-r)!} \\
&= \frac{(k+1)!}{r!((k+1)-r)!} \\
&= \binom{k+1}{r}.
\end{aligned}$$

Hence the formula holds for $n = k+1$ and the result holds by the PMI.

(c) We use the PMI on n. With 1 chord, there can be no intersection points (that is, $m = 0$) and it will cut the circle into $1 + 0 + 1 = 2$ parts trivially, so we have that the formula holds for $n = 1$.

Now suppose the formula holds for some $n = k$. Consider a circle containing $k+1$ chords and m intersection points. Pick any chord from the circle, and let p be the number of intersection points this chord makes with other chords. Now erase this chord from the circle. We are left with k chords and $m - p$ intersection points. By our assumption, there are $k + m - p + 1$ regions that the circle is divided into. Now replace the erased chord. Since it cuts the other chords p times it will divide $p+1$ regions into two new regions, that is $p+1$ regions will be introduced. Hence the total number of regions will be given by $k + m - p + 1 + p + 1 = (k+1) + m + 1$ (see Figure 19), and hence the formula holds for $n = k+1$ as well. Hence by the PMI, the formula is true for all natural numbers n.



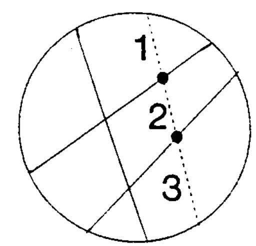


FIGURE 19

(d) If we choose n points on the circumference of a circle, and draw in all possible chords, a maximum amount of regions will result if no three chords are concurrent. Now the amount of possible chords is given by $\binom{n}{2}$, for each chord is uniquely specified by its two end-points. Now if no three chords are concurrent, each possible pair of distinct chords not sharing an endpoint will give rise to a unique internal intersection point. Since each chord is uniquely determined by its two endpoints, the number of such pairs of chords will be the number of ways of choosing 4 distinct end-points from the possible n points, or using (b), $\binom{n}{4}$. Hence by (c), the maximum number of regions S_n is given by $\binom{n}{2} + \binom{n}{4} + 1$.

(8) Firstly, notice that $y \geq x$. To see this, suppose $y < x$. Then $y - x < 0 \implies (y-x)y \leq -1$. Also, $-x^2 \leq -1$, hence $(y-x)y - x^2 \leq -2$, contradicting $|(y-x)y - x^2| = 1$.

Secondly, notice that $y - x \leq x$. To see this, suppose $y - x > x$. Then $(y-x)y - x^2 > xy - x^2 = x(y-x) > x^2 \geq 1$, contradicting $|(y-x)y - x^2| = 1$.

Let $P(n)$ be the statement "Let x and y be distinct natural numbers such that

$$|(y-x)y - x^2| = 1,$$

with $y \leq F_{n+1}$. Then $y - x$, x and y are consecutive terms in the Fibonacci sequence." (Recall F_k is the kth Fibonnaci number introduced in exercises 2(5).) We will show by a combination of the PMI1 and PMI3 that $P(n)$ is true for all $n \geq 2$, hence proving the original problem.

Firstly $P(2)$ is true, since then, if $y \leq F_3 = 2$, since x and y are distinct the only possible result is that $y = 2$, $x = 1$ and $y - x = 1$, yielding the desired result.

Now suppose for some $k \in \mathbb{N}$, $P(m)$ is true for all $m \leq k$. Let x and y be distinct natural numbers such that

$$|(y-x)y - x^2| = 1,$$

with $y \leq F_{k+2}$.

We consider two possible cases:

CASE 1 - $x \leq F_{k+1}$: In this case, let $a = y - x$ and $b = x$. Notice that

$$\begin{aligned} |(b-a)b - a^2| &= |(2x-y)x - (y-x)^2| \\ &= |2x^2 - xy - y^2 - x^2 + 2xy| \\ &= |x^2 + xy - y^2| \\ &= |x^2 - y(y-x)| \\ &= 1. \end{aligned}$$

Hence, since $P(k)$ is true, we may assume $b - a$, a and b are consecutive terms of the Fibonacci sequence, hence $a = y - x$, $b = x$ and $a + b = y$ are consecutive terms of the Fibonacci sequence as well.

CASE 2- $x > F_{k+1}$: In this case, $-x \leq -F_{k+1}$, hence since $y \leq F_{k+2}$, we have $y - x \leq F_{k+2} - F_{k+1} = F_k$. Put $b = y - x$ and $a = 2x - y$. Notice that

$$\begin{aligned} |(b-a)b - a^2| &= |(y - x - 2x + y)(y-x) - (2x-y)^2| \\ &= |2y^2 - 5xy + 3x^2 - 4x^2 + 4xy - y^2| \\ &= |y^2 - xy - x^2| \\ &= |(y-x)y - x^2| \\ &= 1. \end{aligned}$$

Hence, since $P(k-1)$ is true, we have $b - a$, a and b are consecutive terms of the Fibonacci sequence. The sequence will continue as follows: $b - a, a, b, a + b, a + 2b, \ldots$. Hence $b = y - x, a + b = x, a + 2b = y$ are consecutive terms of the Fibonacci sequence.

Since in both cases we have proved that $P(k+1)$ is true, the result holds by the PMI1 and PMI3.