GRAPH THEORY
for the
OLYMPIAD ENTHUSIAST

Graeme West

INTRODUCTION

The South African Mathematical Society has the responsibility for selecting and training teams to represent South Africa in the annual International Mathematical Olympiad (IMO).

The process of finding a team to go to the IMO is a long one. It begins with a nationwide Mathematical Talent Search, in which students are sent sets of problems to solve. Their submissions are marked and returned with comments, full solutions and a further set of problems. The principle behind the Talent Search is straightforward: the more problems you solve, the higher up the ladder you climb and the closer you get to selection.

The best students in the Talent Search are invited to attend Mathematical Camps in which specialised problem–solving skills are taught. The students also write a series of challenging Olympiad–level problem papers, leading to selection of a team of six to go to the IMO.

The bookets in this series cover topics of particular relevance to Mathematical Olympiads. Though their primary purpose is preparing students for the International Mathematical Olympiad, they can with profit be read by all interested high school students who would like to extend their mathematical horizons beyond the confines of the school syllabus. They can also be used by teachers and university mathematicians who are interested in setting up Olympiad training programmes and need ideas on topics to cover and sample Olympiad problems.

Titles in the series published to date are:

No. 1 *The Pigeon-hole Principle*, by Valentin Goranko
No. 2 *Topics in Number Theory*, by Valentin Goranko
No. 3 *Inequalities for the Olympiad Enthusiast*, by Graeme West
No. 4 *Graph Theory for the Olympiad Enthusiast*, by Graeme West
No. 5 *Functional Equations for the Olympiad Enthusiast*, by Graeme West
No. 6 *Mathematical Induction for the Olympiad Enthusiast*, by David Jacobs

Details of the South African Mathematical Society's Mathematical Talent Search may be obtained by writing to

Mathematical Talent Search
Department of Mathematics and Applied Mathematics
University of Cape Town
7700 RONDEBOSCH

The International Mathematical Olympiad Talent Search is sponsored by the Old Mutual.

J H Webb
June 1996

Graph Theory for the Olympiad Enthusiast

Graeme West

Some citizens of Königsberg
Were walking on the strand
Beside the river Pregel
With its seven bridges spanned

'O Euler, come and walk with us',
Those burghers did beseech.
'We'll roam the seven bridges o'er,
And pass but once by each'.

'It can't be done', thus Euler cried.
'Here comes the Q.E.D.
Your islands are but vertices
And four have odd degree'.

From Königsberg to König's book
So runs the graphic tale
And still it grows more colorful
In Michigan, and Yale.

Blanche Descartes, The Expanding Universe

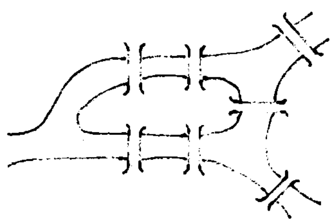
This booklet is intended as an introduction to elementary graph theory for the serious mathematics student participating in the International Mathematics Olympiad training programme. While the theory that is developed in these pages is elementary, some of the problems that arise can be quite challenging, so it is hardly surprising that graph theory problems often feature in mathematics competitions.

Each piece of theory is followed by some exercises which start with some that are designed for consolidation of concepts, and end with problems of an olympiad standard.

1 The Königsberg Bridge Problem

In the early 1700's the citizens of the city of Königsberg in Eastern Prussia entertained themselves with the following problem :-

Example 1.1 *A map of Königsberg and the river Pregel.*

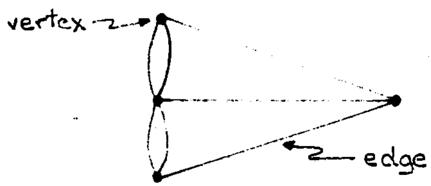


They tried to find a route on their Sunday walks that crossed each bridge exactly once and returned them to their starting point. Try as they might they could find no such route, and they began to believe that the task was impossible.

Leonhard Euler showed in 1736 that there is no possible solution to the Königsberg bridge problem when he published an article called 'The solution of a problem relating to the geometry of position.'

With this article graph theory was born.

We treat the four land areas as single points which are called *vertices.* Singular : *vertex.*



The bridges are *edges* that join the vertices. A collection of vertices and edges is called a *graph.* A graph is sometimes denoted by the letter $\mathcal{G}$. We always accept that there are finitely many vertices in the graph $\mathcal{G}$.

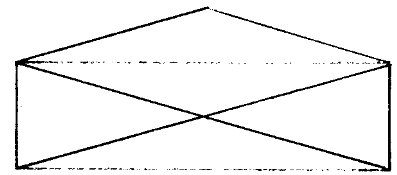
A journey along some edges, passing through some vertices, is called a *path.* The *length* of a path is the number of edges passed over when travelling along the path. A path that ends at the same vertex as where it starts is called a *cycle.* The length of a cycle is the same as the number of vertices passed through, counting the start/finish vertex once only.

passes over all of the edges of the above graph exactly once.'

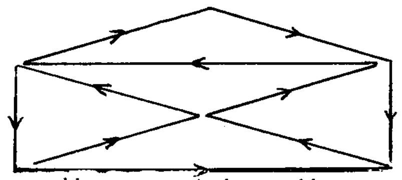
A path that passes over all the edges of a graph exactly once is called an *Eulerian path* in honour of Euler who solved the bridge problem. Of course, if it ends where it started, it is called an *Eulerian cycle.*

Now for another example :-

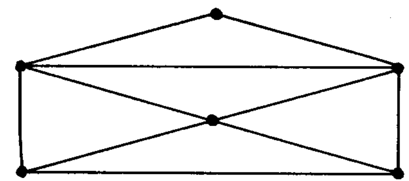
Example 1.2 *Can you draw the picture of the envelope without lifting your pencil from the paper?.*



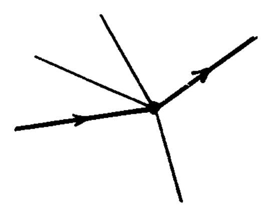
Well there are many solutions. Here is one :-



This is just a graph theory problem : an equivalent problem is to find an Eulerian path for the following graph :-



Now we are ready to do some mathematical reasoning. When we pass over an edge, enter a vertex, and leave it again, we have 'used up' two of the edges that touch that vertex.



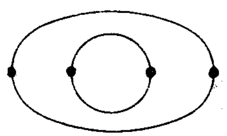
This occurs each time we enter and leave a vertex.

The number of edges that touch a vertex is called the *degree* of that vertex. What we see is that if an Eulerian cycle can be achieved then all vertices will have to have even degree. Or will they? What about the starting vertex? We only use up one degree when we leave the starting vertex. But we use another one degree when returning to it at the end, thus making an even number of degrees for that vertex.

Hence : if an Eulerian cycle exists then all vertices have even degree. So ... there is no solution to the Königsberg bridge problem!

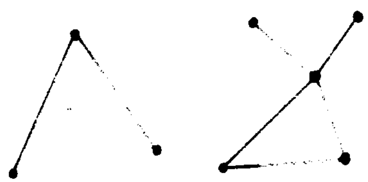
What about the envelope problem? It has two vertices of odd degree so an Eulerian cycle cannot be constructed. But an Eulerian path has been constructed. We see that if an Eulerian path exists then either none or two of the vertices can have odd degree. If two of the vertices have odd degree, then the path must start at one of these two vertices and finish at the other.

We now want to turn all these ideas around and ask : if all the vertices of a graph have even degree then does an Eulerian cycle exist? The answer is no for a very trivial reason. Look at



This is an example of a *disconnected* graph.

A graph is *connected* if there exists some path from every vertex to every other vertex in the graph. As another example, the following graph $\mathcal{G}$ is obviously not connected



but what is hopefully just as obvious is that it is made up of two 'connected pieces'. More formally, we can partition the vertices and edges of $\mathcal{G}$ into two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ each of which are connected. $\mathcal{G}_1$ and $\mathcal{G}_2$ are called the *connected components* of $\mathcal{G}$.

In general, every graph can be broken up into finitely many connected components. Very often when proving results about graphs we can assume the graph is connected because it would be enough to work with the connected components of the original graph.

Theorem 1.3 *Suppose $\mathcal{G}$ is a connected graph in which every vertex is of even degree. Then there exists an Eulerian cycle.*

Proof: We see that $\mathcal{G}$ can be split into disjoint cycles : cycles that have no edge in common. Start at any vertex and then start wandering, deleting edges as they are used. Because every vertex has even degree it is impossible to get stuck at a vertex, and because there are finitely many vertices, we must eventually return to where we started. Thus we construct a cycle. Remove this cycle. Repeat the process again and again on what remains of the graph until there is nothing left. (Note that the graph may become disconnected by performing this process. That doesn't matter, we just perform this process on each of the remaining components of the graph.)

Now that we have split the graph into cycles we put them back together again. Suppose the graph has been split up into N cycles. Choose any two cycles which have a common vertex. We can join these two cycles together - journey on the one cycle until you reach the common vertex, then detour on the other cycle, and then complete the first cycle. Now the two cycles have been replaced by one, and so there are now $N-1$ cycles in total. Repeat until there is just one cycle remaining - which is then Eulerian. ■

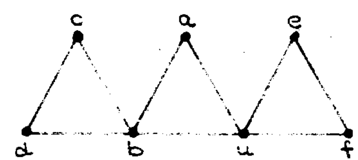
Where is the connectedness of the graph needed in the proof?

Remark 1.4 Given a connected graph which has all vertices of even degree, the following algorithm gives a simple procedure for constructing an Eulerian cycle. This is called *Fleaury's algorithm.*

Choose any starting vertex. Then :-

1. Travel along any edge, with the proviso that if we were to delete that edge then the graph would not become disconnected.
2. Delete the edge.
3. Repeat.

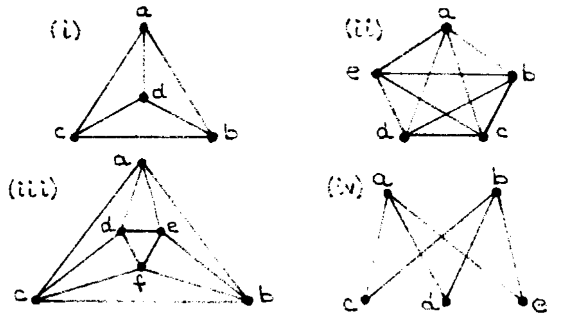
Example 1.5 *Use Fleaury's algorithm to construct an Eulerian cycle, that starts at u, for the following graph:-*



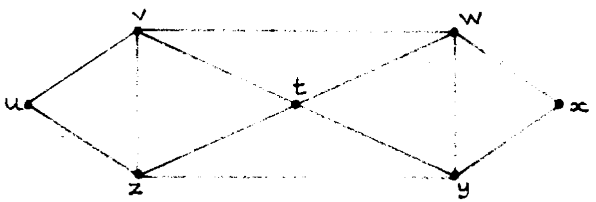
Starting at u, we may choose the edge ua, followed by ab. Erasing these edges (and the vertex a) gives us a situation where we cannot use the edge bu since it would make the graph disconnected, and so we choose the edge bc, followed by cd and db. Then we traverse bu. Traversing the cycle $uefu$ completes the Eulerian cycle.

1.1 Exercises : the Königsberg Bridge Problem

1. Find, if possible, an Eulerian cycle or Eulerian path in the following graphs.



2. Use Fleaury's algorithm to construct an Eulerian cycle for the following graph:-



3. In the country of Jetlaggia it is possible to travel by air between any two of the main cities; if there is not a direct flight there is at least an indirect flight passing through other cities on the way. A path is an air route between two different cities that passes through no intermediate city more than once. The length of a path is the total number of cities on it, counting its endpoint but not its starting point.

   Let M be the maximum of all path lengths in Jetlaggia. Prove that any two paths of length M must have at least one city in common.

   (Australian MO Interstate Final (Senior Division) 1989, Question 1)

2 Standard results and examples

From now on we assume that $\mathcal{G}$ is a graph which has finitely many vertices and that there is at most one edge connecting any two vertices. This is a standard convention and many books include this property in their definition - as we shall see this is quite a natural convention. But note that under this convention the Königsberg bridge 'graph' is not a graph!

Proposition 2.1 *Suppose $\mathcal{G}$ is any graph.*

(a) The sum of the degrees of the vertices of a graph is twice the number of edges in the graph.

(b) There are an even number of vertices that have odd degree.

Proof: (a) Any edge contributes two degrees to the graph : one to the vertex at one end of the edge and one to the vertex at the other end.

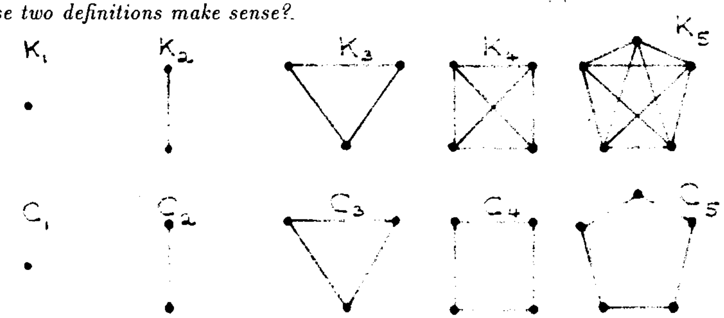
(b) The total number of degrees is even. Subtract from this all of the degrees contributed by vertices with an even degree. What remains is an even number, and it is the total of the degrees contributed by the vertices with an odd degree. The only way a sum of odd numbers can be even is if there is an even number of them. ■

We are now going to see some of the standard graphs.

Definition 2.2 *(a) The graph with n vertices that has an edge between every two vertices is called the complete graph on n vertices, and denoted K_n.*

(b) The connected graph with n vertices where each vertex has degree two is called the cyclic graph on n vertices, and denoted C_n.

Do these two definitions make sense?

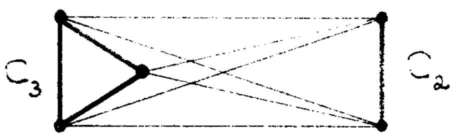


It is clear that K_n has $\binom{n}{2}$ edges and that the degree of each vertex is $n-1$. There are n edges in the graph C_n.

Definition 2.3 *Suppose we have two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Then we can create a new graph as follows : maintain the existing edges between the vertices of $\mathcal{G}_1$ and between the vertices of $\mathcal{G}_2$, and add an edge between every vertex of $\mathcal{G}_1$ and every vertex of $\mathcal{G}_2$.*

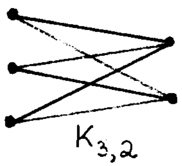
We will call this the complete coupling of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

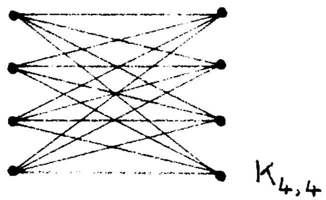
Examples 2.4 (a)



The complete coupling of C_3 *and* C_2.

(b) Suppose we took two very boring graphs : one with n vertices and one with m vertices and no edges in either. The complete coupling of these two graphs is denoted $K_{n,m}$ and is called the complete bipartite graph on n, m vertices. We will see more of this later.





2.1 Exercises : Standard results and examples

1. Is there a graph with 5 vertices where the degrees of the vertices are

   (a) 1,2,3,4,5?

   (b) 0,1,2,3,4?

   (c) 1,1,2,2,3?

2. Suppose we have $n \geq 2$ points in the plane, some of which are connected by line segments. Prove that there are two points which are the ends of the same number of line segments. How many points can there be which are the ends of an odd number of line segments?

3. At a business meeting some of the businesspeople shake hands. Show that there have to be two people who shake hands the same number of times.

4. Can you find a group of 7 people where each knows exactly 3 others?

5. For which of the following graphs can one find an Eulerian cycle?

   (a) K_n

   (b) $K_{n,m}$

   (c) C_n

6. Find a formula for the number of edges in the complete coupling of two graphs that involves information about the original two graphs.

7. Suppose there is a party attended by 5 married couples. Various handshakes take place, but of course no couples shake hands (they know each other already, hopefully!) and no pair of people shake hands more than once. Afterwards, the host asks the other 9 people how many times they shook hands. To his surprise, the answers were all different.

   How many times did the host, and his wife, shake hands?

8. Suppose $\mathcal{G}$ is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges the greatest common divisor of the integers labelling those edges is equal to 1.

   (IMO 1991, Question 4)

3 Colouring the edges of a graph

We are now going to consider the problem of colouring the edges of certain graphs. We will be given a certain graph and an instruction to shade the edges of the graph in a particular way. The problem will be then to logically determine whether or not it is possible to shade the edges in that way.

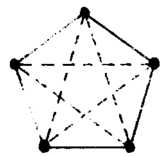
We will be most interested in *monochromatic cycles.* A monochromatic cycle is a cycle whose edges are all coloured the same colour. A monochromatic triangle is a monochromatic cycle of length three. A monochromatic odd/even cycle is a monochromatic cycle of odd/even length.

Lets start straight away with perhaps the most well known result in graph theory :

Theorem 3.1 *If we shade the edges of* K_6 *with two colours, red and blue say, then there must be a monochromatic triangle.*

Proof: Take one of the vertices, v_0, arbitrarily. This vertex has degree 5, and so of the 5 edges from that vertex, by the Pigeonhole Principle at least three must be of one colour. Without loss of generality this colour is red. Suppose the red edges end at v_1, v_2, v_3. If any one of the edges between v_1 and v_2, v_2 and v_3 or v_3 and v_1 were red, then we would have a red triangle. If none of them were red, then they would all have to be blue, and then $v_1v_2v_3$ would form a blue triangle. ■

I think such an elegant and simple argument deserves to be famous. Now we should note that if $n \geq 6$ then K_n will have a monochromatic triangle when shaded with two colours, since it has K_6 as a subgraph. On the other hand, we can shade K_5 with two colours in such a way that there is no monochromatic triangle.



(We will see in §4 that K_5 will always have a monochromatic odd cycle.)

Now suppose we have three colours. We want to find the smallest value of n such that K_n is guaranteed to have a monochromatic triangle. We claim that K_{17} is guaranteed to have a monochromatic triangle. (Its going to be obvious immediately where 17 came from.)

Choose any one of the vertices. There are 16 edges from that vertex, and so by the Pigeonhole Principle six of them must be one of the three colours, say red. Look at the vertices at which these six edges end. If any of the edges between these six vertices are shaded red, then we have a monochromatic red triangle. If none of them are red, then we have a subgraph K_6 which is shaded only in the other two colours. Then by Theorem 3.1 we would have to have a monochromatic triangle there.

What remains to be shown is that 17 is the optimal number, that is, it is possible to shade K_{16} with three colours and no monochromatic triangle. (This isn't obvious.)

3.1 Exercises : Colouring the edges of a graph

1. Find the smallest(?) value of n for which whenever the edges of K_n are coloured with 4 colours then there exists at least one monochromatic triangle.

2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces is given. Each side of the two pentagons and each of the line segments A_iB_j where $1 \leq i, j \leq 5$ is coloured either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been coloured has two sides of a different colour. Show that all 10 sides of the top and bottom faces are the same colour.

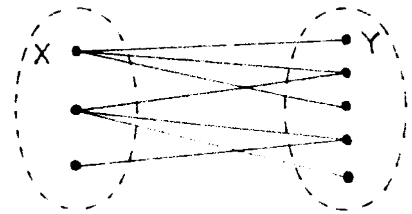
   (IMO 1979, Question 2)

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

   (IMO 1992, Question 3)

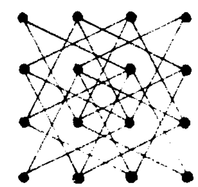
4 Bipartite graphs

Definition 4.1 *A bipartite graph $\mathcal{G}$ is one where the vertices can be divided into two disjoint sets X and Y so that every edge of the graph $\mathcal{G}$ connects a vertex in X to a vertex in Y.*



Example 4.2 $K_{m,n}$ is by definition a bipartite graph.

Example 4.3 This next example assumes a small amount of familiarity with the game of chess. Suppose we have a chessboard (actually, this does not have to be an 8 by 8 board - all we need is that the squares are coloured in black and white the way a chessboard usually is). We now think about the way a knight moves on a chessboard : two squares either horizontally/vertically and then one square vertically/horizontally. Then we have a graph : the vertices are the squares of the chessboard, and an edge is drawn between vertices iff a knight could move from the one square (vertex) to the other square (vertex). This is called the *Knight's tour graph.* Here, for example, is the knight's tour graph on a 4 by 4 board :-



Now notice that if a knight is on a white square then when it moves it moves to a black square, and vice versa.

So the Knight's tour graph is bipartite : we let X be the set of all white squares and let Y be the set of all black squares.

Notice that if a knight starts on a white square then after an even number of moves it will be on a white square and after an odd number of moves it will be on a black square. In particular, any cycle must be of even length. This is an example of a more general phenomenon which is dealt with in the following theorem.

Theorem 4.4 *A graph is bipartite* $\iff$ *all its cycles are of even length.*

Proof: A little thought will convince you that for a graph to be bipartite it will have to have all its cycles of even length.

Now for the converse : it suffices to consider connected graphs. (Why?) Suppose the graph has all its cycles of even length. Then all paths from a vertex a to a vertex b will be of the same parity; for otherwise an odd-lengthed and a different even-lengthed journey from a to b implies an odd-lengthed cycle : from a, to b, and back to a.

Now select a vertex a arbitrarily, and divide the vertices into two sets X and Y : let X be those vertices that are an even distance from a (this will include a, since a is 0 distance from a, and 0 is even), and let Y be those vertices that are an odd distance from a.

You can check this gives the desired partition of the graph. ∎

We now have the following very interesting and useful corollary which continues our discussion of the colouring of graphs.

Corollary 4.5 *If* $m > 2^n$ *and the edges of* K_m *are coloured with* n *colours then there exists a monochromatic cycle of odd length.*

Proof: The proof is by induction; of course it is obvious for $n = 1$.

Suppose the edges of K_m are coloured n different colours. Concentrate on one of the colours, say red. If the subgraph of red edges has an odd cycle then there is nothing to do, so suppose all monochromatic red cycles are of even length. Then the subgraph of the red edges is bipartite by Theorem 4.4.

Form the bipartite partition of all the m vertices as determined by the red graph; say the vertices are divided into sets X and Y. (Thus all the red edges connect a vertex in X with a vertex in Y.) One of the sets X and Y has $p > 2^{n-1}$ points, say it is X. Consider the subgraph (all the colours now) with the points of X as vertices; of course this is a copy of K_p. But by construction none of the edges in this K_p are red.

Thus we have the graph K_p which is coloured with $n - 1$ colours, and $p > 2^{n-1}$. By induction there is an monochromatic cycle of odd length in K_p. Putting everything back together, we thus have an odd monochromatic circuit in K_m (which happens to be one of the other colours besides red). ∎

Example 4.6 We have seen previously that for $n \geq 6$, K_n must have a monochromatic triangle when the edges are shaded with two colours, and that this does not hold if $n \leq 5$. However, K_5 must have a monochromatic cycle of odd length, since $5 > 4 = 2^2$.

Example 4.7 Suppose 3 airlines serve 10 cities in such a way that there is a direct service with one or more of the airlines between each city, and all journeys have corresponding return journeys.

Then at least one of the airlines offers a round trip with an odd number of landings, since $10 > 8 = 2^3$.

4.1 Exercises : Bipartite graphs

1. Suppose 3 airlines serve 8 cities in such a way that there is a direct service with one or more of the airlines between each city, and all journeys have corresponding return journeys. Is it possible that all of the round trips provided by the 3 airlines have an even number of landings?

2. Ten cities are served by two airlines in such a way that there exists a direct service between any two of the cities and all airline schedules are both ways.

   Show that at least one of the airlines can offer two disjoint round trips each containing an odd number of landings.

   (Submitted for the IMO 1990)

5 Hamiltonian cycles

Definition 5.1 *A cycle in a graph* $\mathcal{G}$ *is said to be Hamiltonian if it passes through each vertex of* $\mathcal{G}$ *exactly once.*

It is clear that K_n has a Hamiltonian cycle for $n \geq 3$. Also, C_n has a Hamiltonian cycle for $n \geq 3$ (and then there is nothing left).

It is interesting to ask when one can take a graph and decompose it into disjoint Hamiltonian cycles. That means : remove a Hamiltonian cycle, remove another Hamiltonian cycle, ..., until there is nothing left.

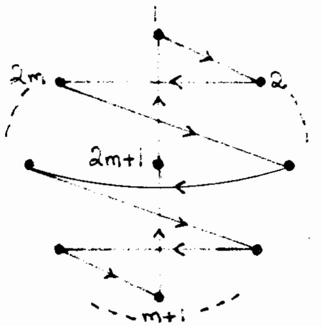
So in a very trivial manner C_n is decomposed into disjoint Hamiltonian cycles, since you remove one cycle and you're already finished. What about K_5? You can check for yourself that K_5 can be decomposed into two Hamiltonian cycles. What about K_4? Try as you might you won't succeed, and there is an elegant way of arguing this. Try to work it out before proceeding to the proof of the next theorem.

Theorem 5.2 *(Lucas)*
K_n is the union of Hamiltonian cycles iff n is odd.

Proof: Consider the graph K_n. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges and each Hamiltonian cycle will use n of them, so we need to construct $\frac{n-1}{2}$ cycles. In particular n needs to be odd.

Now suppose $n = 2m + 1$ is indeed odd. We need to construct m disjoint Hamiltonian cycles. Let $2m$ of the vertices be represented by a regular $2m$-gon with one vertex facing 'north'. Label the vertices $1, 2, \ldots, 2m$, starting at 'north' and moving clockwise. Finally let the $2m + 1$-th vertex be in the middle of the gon.

Now consider the cycle :



This is a Hamiltonian cycle.

Now 'leave' the edges in place, but 'rotate' the vertices around the centre (say clockwise) through an angle of $\frac{360°}{2m}$. The edges trace out a new Hamiltonian cycle. We repeat this process m times (until just before 'north' and 'south' have swopped).

Simple geometry shows that no two of the cycles have a common edge, and hence the graph is decomposed into Hamiltonian cycles. ∎

Example 5.3 The graph K_9 is decomposed into the following four disjoint Hamiltonian cycles by the method of Lucas :-

9 1 2 8 3 7 4 6 5 9,

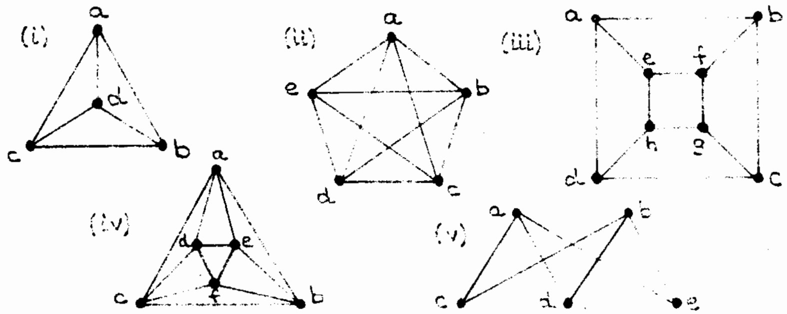
9 8 1 7 2 6 3 5 4 9,

9 7 8 6 1 5 2 4 3 9,

9 6 7 5 8 4 1 3 2 9.

5.1 Exercises : Hamiltonian cycles

1. Find, if possible, a Hamiltonian cycle in the following graphs.



2. Can a knight perform a Hamiltonian cycle on a 13 by 15 chessboard?

3. Decompose K_7 into disjoint Hamiltonian cycles.

4. Let n be a positive integer and $A_1, A_2, \cdots, A_{2n+1}$ subsets of a set B. Suppose that

   (a) each A_i has exactly $2n$ elements,

   (b) each $A_i \cap A_j$ $(1 \leq i < j \leq 2n+1)$ contains exactly one element, and

   (c) every element of B belongs to at least two of the A_i.

   For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that each A_i has 0 assigned to exactly n of its elements?

   (IMO 1988, Question 2)

6 Solutions

6.1 Solutions : The Königsberg Bridge Problem

1. (i) No Eulerian path or cycle.

   (ii) *abcdeacebda* is an Eulerian cycle.

   (iii) *abcadcfbefdea* is an Eulerian cycle.

   (iv) No Eulerian cycle; *acbeadb* is an Eulerian path.

2. Remove uv and vz. We cannot use the edge zu (this is a bridge, i.e. an edge which, if removed, would disconnect the graph). So we must use either zt or zy. For example, we transverse $ztvwy$. Now yz is illegal since this is also a bridge, so we transverse $ytwxy$, and then return (since there is no alternative) by the bridges yz and zu. So a possible Eulerian trail is

$$uvztvwytwxyzu$$

3. The question is simply asking : if $\mathcal{G}$ is a connected graph then any two paths of maximum length must have a vertex in common.

   Let $x_0, \ldots, x_M$ and $y_0, \ldots, y_M$ be two paths each of length M, and suppose these paths have no cities in common.

   There must be, by the hypothesis of the question, a path $x_i, \ldots, y_j$ connecting a city x_i and a city y_j, and not passing through any other x_k and y_l.

   One of the subpaths $x_0, \ldots, x_i$ and $x_M, \ldots, x_i$ is of length at least $\frac{M}{2}$; suppose it is $x_0, \ldots, x_i$. Likewise suppose $y_0, \ldots, y_j$ is of length at least $\frac{M}{2}$. Then the path

$$x_0, \ldots, x_i, \ldots, y_j, \ldots, y_0$$

   is of length at least $M+1$, contradicting the fact that M was the maximum path length.

6.2 Solutions : Standard results and examples

1. The answer is no in all three cases : (a) if there are n vertices then the rules of the game imply that any degree is at most $n-1$; (b) we can't have one vertex joined to all the others and another vertex joined to none; (c) we can't have three vertices of odd degree.

2. Consider a graph where the points are the vertices and the line segments are the edges. There are n vertices and the possible values for the degrees is $0, 1, \ldots, n-1$. Now 0 and $n-1$ cannot both occur, so in actual fact there are only $n-1$ possible values for the degrees of the n vertices. So two of the vertices will have the same degree.

3. This is the same question as the one previously.

4. Consider a graph of 7 vertices, representing the people, and let edges be drawn between people who know each other. If the scenario were possible, then each vertex would be of degree 3 i.e. there would be an odd number of vertices with odd degree. This is known to be impossible.

5. All three of these graphs are connected, so we only need examine their degrees.

   (a) the degree of each vertex is $n-1$, so the graph has an Eulerian cycle iff n is odd.

   (b) The degrees of the vertices are m or n, so the graph has an Eulerian cycle iff both m and n are even.

   (c) For $n \geq 3$ the degree of any vertex is two, and so an Eulerian cycle exists.

6. Suppose the two graphs are $\mathcal{G}_1$ and $\mathcal{G}_2$ and they have v_i vertices and e_i edges $(i = 1, 2)$. Then the number of edges in the complete coupling is

$$e_1 + e_2 + v_1 v_2$$

7. We consider a graph with ten vertices where the handshakes are the edges. The degrees of the ten vertices are

$$8, 7, 6, 5, 4, 3, 2, 1, 0, h$$

   where h is the degree of the host. It is now clear that the person of degree 8 is married to the person of degree 0. We remove these two from the party, as if they had never been there and as if the 8 handshakes had never taken place. We have then the degrees of the eight remaining vertices are

$$6, 5, 4, 3, 2, 1, 0, h-1$$

   and so again we see that '6' and '0' are married. We remove them again. We repeat this process until we get

$$0, h-4$$

   which are the wife of the host and the host. If we trace these people back to the original party, we see they both have degree 4.

8. Start at some vertex v_0. Imagine yourself walking along distinct edges of the graph, numbering them $1, 2, \ldots$ as you encounter them, until you cannot go any further without reusing an edge.

   If there are edges which are not numbered, one of them has a vertex which has been visited, since $\mathcal{G}$ is connected. Starting with this vertex, continue to walk along unused edges, resuming the numbering where you left off, until once again you can go no further. Repeat this procedure until all the edges are numbered.

We now prove that the numbering satisfies the stated condition that at each vertex belonging to two or more edges the GCD of the numbers of the edges meeting at that vertex is 1. Let v be such a vertex. If $v = v_0$ i.e. v is the starting point, then one of the edges meeting at v is labelled one, and so the GCD at v is 1. If $v \neq v_0$, suppose the first time you encountered v on the walk was at the end of the edge labelled r. At that time there were one or more unused edges at v, one of which was then labelled $r+1$. The GCD of any set containing both r and $r+1$ is 1.

6.3 Solutions : Colouring the edges of a graph

1. You should get 66 vertices. The argument is identical to the case of two or three colours, but it's not at all obvious that 66 is optimal.

2. Suppose the top edges are not all the same colour. We may suppose then that edge A_1A_2 is red and edge A_2A_3 is green. At least three of the edges A_2B_j $(1 \leq j \leq 5)$ are the same colour, say red. Label these edges A_2B_r, A_2B_s, A_2B_t. Then (here is the important bit) at least one of B_rB_s, B_rB_t, B_sB_t is a base edge. Suppose it is B_rB_s. Clearly then this edge must be green. Now A_1B_r and A_1B_s must also be green, for otherwise we would have $A_1A_2B_r$ or $A_1A_2B_s$ as red triangles. Therefore $A_1B_rB_s$ is a green triangle. This contradiction implies that A_1A_2 and A_2A_3 have the same colour and similarly that all the edges of the base have the same colour.

   Now suppose the top edges are all red and the bottom edges are all green. If three green edges join A_1 to the base, two of them must terminate on adjacent vertices B_r, B_s of the base. Then $A_1B_rB_s$ is a green triangle, a contradiction. Hence at least three red edges join A_1 to the bottom. Similarly, at least three red edges join A_2 to the bottom. Since we now have six red edges, at least two of them must terminate in the same vertex, giving a red triangle : the final contradiction.

3. (This superb problem featured on the first paper at the IMO in Moscow, 1992. As such it formed part of a baptism of fire for the first South African team to attend the Olympiad. It is unquestionably very difficult, and was spoken of very highly by the Russian officials.)

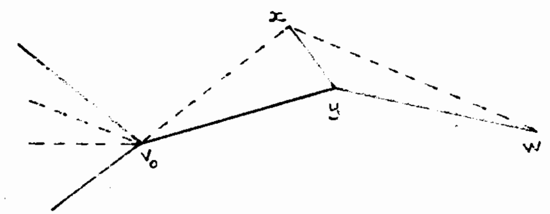
   What we have is the graph K_9 where the edges are coloured either red or blue or left uncoloured (alternatively, one can think that some edges have been removed). If one edge is removed, there remains within the graph a shaded copy of K_8, which must have a monochromatic triangle. Similarly if two edges are removed, there remains within the graph a shaded copy of K_7; and if three edges are removed, there remains within the graph a shaded copy of K_6. So if up to three edges are removed then a monochromatic triangle is guaranteed. Now there are in total 36 edges in the graph, so if $n = 36 - 3 = 33$ then a monochromatic triangle is guaranteed.

If we removed four edges, then we would be left with a copy of K_5, which can be shaded without monochromatic triangles. So it is feasible that if we remove four edges from K_9 then the graph can be shaded without monochromatic triangles. To complete the problem, all we need do is construct an example showing that this is indeed possible. And as already hinted at, we should start from a shaded copy of K_5 that has no monochromatic triangles and build it up (that is, add vertices and edges) until we arrive at the required graph.

To achieve this, we can formulate the following algorithm : suppose $\mathcal{G}$ is any graph shaded in two (or more) colours and without monochromatic triangles. Then we can formulate the following method for forming a new graph $\mathcal{G}'$ with one extra vertex and no monochromatic triangles and an 'optimal' increase in the number of edges.

(a) Choose a vertex v_0 of maximal degree.

(b) Add a vertex w, and join w to all the vertices x of $\mathcal{G}$ that v_0 is joined to, colouring the edge wx the same colour as that of v_0x.



Now given any two points x, y the triangle wxy is not monochromatic because the triangle v_0xy is not. So there are no monochromatic triangles in the new graph $\mathcal{G}'$.

This algorithm can be repeated as often as is necessary, adding further points.

We note that

(a) the degree of w is the degree of v_0;

(b) the degree of v_0 does not change;

(c) all vertices that v_0 is connected to increase in degree by 1.

We apply this algorithm to K_5 which is coloured without monochromatic triangles, and repeat four times. We count the degrees of each of the vertices during the process.

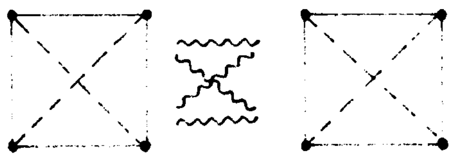
Vertices	Degrees	$\sum$ degrees	# Edges
5	44444	20	10
6	455554	28	14
7	5566655	38	19
8	66677666	50	25
9	777787777	64	32

So we have a graph with 9 vertices, 32 edges, and no monochromatic triangle - as required.

We leave it as a further exercise for you to show that if instead of 9 vertices we had considered 6,7,8 or 10 vertices, the same method would have given a full solution ($n = 15$,20,26 or 41). However, in the case of 11 vertices, a consideration of K_6 tells us that $n = 50$ guarantees a monochromatic triangle, while the algorithm produces a graph with 48 (NO, NOT 49!) edges. So what happens here : is the answer 50 or is it 49?

6.4 Solutions : Bipartite graphs

1. We take two copies of K_4 shaded in two colours without monochromatic odd cycles and then the complete coupling of these graphs, the 'coupling edges' being all of the third colour.



2. Phrased in the language of graph theory : if the edges of K_{10} are shaded either red or green then we want to show that there exist two red or two green disjoint cycles of odd length.

   By considering a subgraph K_6 of K_{10} we can find a monochromatic triangle (which of course is of odd length). Removing these three points, we are left with the graph K_7. We can repeat the above process and find another monochromatic triangle. If these two triangles are of the same colour then we are already finished, so we suppose that one is red (on vertices u, v, w) and one is green (on vertices x, y, x). Label the other vertices a, b, c, d.

   We now examine the edges between u, v, w and x, y, z.

   Two of the edges from u to x, y, z must be of the same colour. Suppose they are to x and to y. If this colour were green then we would have a green triangle uxy, a red triangle uvw and then 5 unused points : z, a, b, c, d. By applying Corollary 4.5 with $m = 5$ and $n = 2$ we have that on z, a, b, c, d there is a monochromatic odd cycle. Combined with the triangle of the same colour listed above we would have two disjoint odd cycles of the same colour.

   Thus ux and uy must be red. For the same reasons at least two of the edges from w to x, y, z must be red, and at least one of these must be to x or y. Say it is to x. Then we would have a red triangle uwx, a green triangle xyz and 5 further points v, a, b, c, d. As above we have two disjoint odd cycles of the same colour.

6.5 Solutions : Hamiltonian cycles

1. (i) $abcda$ is a Hamiltonian cycle.

   (ii) $abcdea$ is a Hamiltonian cycle.

   (iii) $abcdhgfea$ is a Hamiltonian cycle.

   (iv) $abcdfea$ is a Hamiltonian cycle.

   (v) There is no Hamiltonian cycle.

2. There are an odd number of squares on this board. That means that the knight must make an odd number of moves in its Hamiltonian cycle. But since the graph is bipartite, all cycles are of even length. Therefore there is no Hamiltonian cycle.

3. 16253471; 65142376; 54631275.

4. We first note that it fact every element of B belongs to exactly two of the sets A_i. Since $|A_i| = 2n$ and $|A_i \cap A_j| = 1$ for $i \neq j$ and $|\{j : i \neq j\}| = 2n$, we have that each A_i is the union of its intersection with each of the other A_j's, and there is a contribution of exactly one element from each such A_j. So if some element belonged to three of the A_i's then the second and third set would both be trying to contribute that element to the first, which is impossible.

   We now consider the graph K_{2n+1} where the sets A_i are the $2n + 1$ vertices and the edge between A_i and A_j is the unique element in $A_i \cap A_j$.

   Now since $2n + 1$ is odd we have that K_{2n+1} is expressed as the disjoint union of Hamiltonian cycles - in fact, n Hamiltonian cycles.

   If n is even then take any $\frac{n}{2}$ of the cycles and label the edges of these cycles 0, the others 1. Then each A_i has 0 assigned to n of its elements.

   Suppose now that n were odd, and suppose that the assignment were possible. Now remove all the edges labelled 0 from the graph. The remaining graph has $2n + 1$ vertices each of degree n, that is, an odd number of vertices of odd degree. This is impossible, so the assignment is impossible if n is odd.