INEQUALITIES
for the
OLYMPIAD ENTHUSIAST

Graeme West

INTRODUCTION

The South African Mathematical Society has the responsibility for selecting and training teams to represent South Africa in the annual International Mathematical Olympiad (IMO).

The process of finding a team to go to the IMO is a long one. It begins with a nationwide Mathematical Talent Search, in which students are sent sets of problems to solve. Their submissions are marked and returned with comments, full solutions and a further set of problems. The principle behind the Talent Search is straightforward: the more problems you solve, the higher up the ladder you climb and the closer you get to selection.

The best students in the Talent Search are invited to attend Mathematical Camps in which specialised problem–solving skills are taught. The students also write a series of challenging Olympiad–level problem papers, leading to selection of a team of six to go to the IMO.

The bookets in this series cover topics of particular relevance to Mathematical Olympiads. Though their primary purpose is preparing students for the International Mathematical Olympiad, they can with profit be read by all interested high school students who would like to extend their mathematical horizons beyond the confines of the school syllabus. They can also be used by teachers and university mathematicians who are interested in setting up Olympiad training programmes and need ideas on topics to cover and sample Olympiad problems.

Titles in the series published to date are:

No. 1 *The Pigeon-hole Principle,* by Valentin Goranko
No. 2 *Topics in Number Theory,* by Valentin Goranko
No. 3 *Inequalities for the Olympiad Enthusiast,* by Graeme West
No. 4 *Graph Theory for the Olympiad Enthusiast,* by Graeme West
No. 5 *Functional Equations for the Olympiad Enthusiast,* by Graeme West
No. 6 *Mathematical Induction for the Olympiad Enthusiast,* by David Jacobs

Details of the South African Mathematical Society's Mathematical Talent Search may be obtained by writing to

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The International Mathematical Olympiad Talent Search is sponsored by the Old Mutual.

J H Webb
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Inequalities for the Olympiad Enthusiast

Graeme West

In this booklet we discuss standard mathematical inequalities that should be in the armoury of the olympiad competitor. In earlier years of the International Mathematics Olympiad, questions using inequalities such as the Cauchy-Schwartz inequality or the Arithmetic-Geometric mean inequality were commonplace. In more recent times, as the standard of questions set at the IMO has undergone a substantial increase, the use of inequalities has become more subtle : competitors are thrown more onto their own resources and are expected to be able to decide for themselves what tool is appropriate in any given situation and to be able utilise that tool correctly. IMO problems have changed from one or two step problems to multistep problems. The aim of this booklet is to make the reader fully competent with one of these possible steps : manipulating with inequalities. On the other hand, routine one or two step problems are of course prevalent, and rightly so, in competitions such as the National Olympiad.

We begin with my personal favourite, the rearrangement inequality. This is a tremendously powerful but very simple inequality which strangely enough is to a large degree unknown. Then we deal with the Arithmetic-Geometric mean inequality, the triangle inequality and the Cauchy-Schwartz inequality. Finally we deal with Jensen's inequality for convex and concave functions. (There are some other inequalities involving other means such as the Harmonic mean, for example, which have been omitted. But such tools I do not regard as standard and moreover I have yet to find a competition problem which needs such a tool for the most natural solution. Nor have we dealt with inequalities that are established, for example, by induction : usually these problems are problems in induction *per se* and not in inequalities.) There are a large number of exercises after each chapter, many from past IMO papers, and comprehensive solutions at the end.

I should point out that for a number of problems more than one method of solution is possible. I have tried to include the problem in the most natural section of the booklet, but comments in this regard are very welcome.

1 The Rearrangement Inequality

The rearrangement inequality is perhaps the most useful of all the inequalities that we can consider, and is very simple to understand.

Consider the numbers $1, 2, \dots, n$. Any rearrangement of these numbers is represented by a unique function σ, called a permutation. If, for example, we arranged the numbers 1,2,3,4 in the order 3,1,2,4 then we would write

$$\sigma(1) = 3,\ \ \sigma(2) = 1,\ \ \sigma(3) = 2,\ \ \sigma(4) = 4$$

Note that if we take $1, 2, 3, \dots, n$ and rearrange them as $n, n-1, \dots, 3, 2, 1$ then we have the permutation with formula $\sigma(i) = n + 1 - i$.

The rearrangement inequality says the following:-

Theorem 1.1 *(The Rearrangement Inequality)*
Suppose $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. Then

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i} \qquad (1)$$

for any permutation σ.

Example 1.2 Suppose $x_1 \geq x_2$ and $y_1 \geq y_2$. Then $x_1y_1 + x_2y_2 \geq x_1y_2 + x_2y_1$

Example 1.3 Suppose $x_1 \geq x_2 \geq x_3$ and $y_1 \geq y_2 \geq y_3$. Then

$$x_1y_1 + x_2y_2 + x_3y_3 \geq x_1y_3 + x_2y_2 + x_3y_1$$

and the other variations, namely $x_1y_1+x_2y_3+x_3y_2$, $x_1y_2+x_2y_1+x_3y_3$, $x_1y_2+x_2y_3+x_3y_1$ and $x_1y_3 + x_2y_1 + x_3y_2$ lie in between these two extreme values.

We will shortly give a formal proof of the rearrangement inequality, but first we consider a fairly convincing intuitive motivation:-

Suppose we have n people of weights $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$ and we have n seats on a see-saw which are $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ distance away from (one side of) the central pivot. Then to maximise the moment on that side of the see-saw, we should put the heaviest person the furthest away, the next heaviest second furthest away, etc. This gives a moment of $\sum_{i=1}^{n} x_i y_i$. To minimise the moment we should arrange the people in the reverse order, and we get a moment of $\sum_{i=1}^{n} x_i y_{n+1-i}$. Any other arrangement σ of the people in the seats will give us a moment of $\sum_{i=1}^{n} x_i y_{\sigma(i)}$, somewhere in between these two extremes.

From this it is evident that if there is equality in the expression

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)}$$

then any pair of people who have changed places in fact have equal mass. Expressed mathematically, this says that the finite sequence $\{y_i\}$ is the same as the sequence $\{y_{\sigma(i)}\}$. Similarly if there is equality in the expression

$$\sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i}$$

then the finite sequence $\{y_{\sigma(i)}\}$ is the same as the sequence $\{y_{n+1-i}\}$.

Proof: (of Theorem 1.1)
If σ is a non-trivial rearrangement then there must exist i, j such that $i > j$ but $\sigma(i) < \sigma(j)$. Therefore $y_{\sigma(i)} \geq y_{\sigma(j)}$, and so

$$x_i y_{\sigma(j)} + x_j y_{\sigma(i)} - \left(x_i y_{\sigma(i)} + x_j y_{\sigma(j)}\right) = (x_j - x_i) \cdot \left(y_{\sigma(i)} - y_{\sigma(j)}\right) \geq 0$$

as each term is positive. Therefore, modifying σ by exchanging $\sigma(i)$ and $\sigma(j)$ increases the value of $\sum_{i=1}^{n} x_i y_{\sigma(i)}$. It then follows that this process will only terminate if the arrangement σ is trivial, and we deduce that

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)}$$

Similarly we deduce that

$$\sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i}$$

■

Example 1.4 Suppose $x, y \in \mathbf{R}$. Then

$$x^2 + y^2 \geq 2xy \qquad (2)$$

(This is in fact equivalent to the statement of the arithmetic-geometric inequality for two variables.)

Example 1.5 Show that if $x, y, z > 0$ then

$$x^3 + y^3 + z^3 \geq 3xyz$$

By symmetry we may suppose $x \geq y \geq z$. Then of course $x^2 \geq y^2 \geq z^2$ and so

$$\begin{aligned} x \cdot x^2 + y \cdot y^2 + z \cdot z^2 &\geq x \cdot y^2 + y \cdot z^2 + z \cdot x^2 \\ &= xy \cdot y + xz \cdot x + yz \cdot z \end{aligned}$$

and $xy \geq xz \geq yz$, so

$$\begin{aligned} xy \cdot y + xz \cdot x + yz \cdot z &\geq xy \cdot z + xz \cdot y + yz \cdot x \\ &= 3xyz \end{aligned}$$

Note 1.6 In (1) we required that the x_i's and y_i's be arranged in decreasing order. This is not strictly necessary: all that is required is that the x_i's and y_i's be arranged in the same order with respect to size. This idea is best illustrated by means of the following example:

Example 1.7 Show that if $x, y, z > 0$ then

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$$

Note that the expression is not symmetric in x, y, z so we cannot simply suppose, for example, that $x \geq y \geq z$. Instead we note that if x, y, z are is a certain size order then $\frac{1}{z}, \frac{1}{y}, \frac{1}{x}$ will be in the same order. So $\frac{x}{z} + \frac{y}{y} + \frac{z}{x}$ is the largest possible sum, and $\frac{x}{x} + \frac{y}{y} + \frac{z}{z} = 3$ is the smallest. In particular, $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$.

Example 1.8 Show that if $x, y, z > 0$ then

$$\frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} \geq 3$$

The expression is symmetric in the variables x, y, z so we may suppose without loss of generality that $x \geq y \geq z > 0$. Then

$$xy \geq xz \geq yz$$

and

$$\frac{1}{z^2} \geq \frac{1}{y^2} \geq \frac{1}{x^2}$$

Thus

$$\frac{xy}{z^2} + \frac{xz}{y^2} + \frac{yz}{x^2} \geq \frac{xy}{y^2} + \frac{xz}{x^2} + \frac{yz}{z^2} = \frac{x}{y} + \frac{z}{x} + \frac{y}{z}$$

and so the result follows from the previous example.

1.1 Exercises : the rearrangement inequality

To do some of these exercises you will need to know about the 'Ravi substitution' : when a, b, c are the sides of a triangle, we can take

$$a = x + y, \quad b = y + z, \quad c = z + x$$

for some $x, y, z > 0$. The values of x, y, z are determined by constructing the incircle of the triangle. In solving inequalities, this is a very useful substitution as we no longer need worry about all the $a + b > c$ stuff : it's built right in.

1. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$2 \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

   Can equality hold on either side?

   (Indian National MO 1989, Question 5)

2. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc$$

   (IMO 1964, Question 2)

3. Show that if $a, b, c > 0$ then

$$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

4. Show that if $a, b, c > 0$ then

$$a + b + c \leq \frac{a^2 + b^2}{2c} + \frac{b^2 + c^2}{2a} + \frac{c^2 + a^2}{2b} \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}$$

5. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0$$

   Determine when equality occurs.

   (IMO 1983, Question 6)

6. Show that if $0 < x, y, z$ then

$$x^3 + y^3 + z^3 \geq y^2z + z^2x + x^2y$$

   (British MO 1981, Question 3(a))

7. Let $\{a_k\}$ be a sequence of distinct positive integers. Prove that for all $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}$$

   (IMO 1978, Question 5)

8. **The Chebyshev inequality**
Prove that if $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ then

$$n \cdot \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \tag{3}$$

9. Prove that if $x, y > 1$ then

$$n \cdot \frac{x^n y^n - 1}{xy - 1} \geq \frac{x^n - 1}{x - 1} \cdot \frac{y^n - 1}{y - 1}$$

2 The Arithmetic-Geometric Mean Inequality

Definition 2.1 *Suppose $x_1, x_2, \ldots, x_n \geq 0$. Then*

(a) the quantity

$$\frac{x_1 + x_2 + \cdots + x_n}{n}$$

is called the arithmetic mean of the numbers $x_1, x_2, \ldots, x_n$;

(b) the quantity

$$\sqrt[n]{x_1 x_2 \cdots x_n}$$

is called the geometric mean of the numbers $x_1, x_2, \ldots, x_n$

Suppose $x, y \geq 0$. Then the arithmetic mean of x and y is $\frac{x+y}{2}$ and the geometric mean is $\sqrt{xy}$. We can actually show that no matter the values of x and y, we always have that the arithmetic mean is greater than or equal to the geometric mean :

It is clear that $\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$. Now multiply out and transfer terms and we get

$$x + y \geq 2\sqrt{x}\sqrt{y}$$

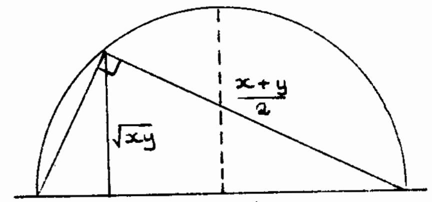
which simplifies to

$$\frac{x+y}{2} \geq \sqrt{xy}$$

This is called the arithmetic-geometric mean inequality for two variables. We can ask in addition : when could we have equality here? Well it is clear that that would only happen if $\sqrt{x} - \sqrt{y} = 0$ i.e. if and only if $x = y$. Expressed differently : the arithmetic-geometric mean inequality is an equality if and only if the two numbers featured are equal.

There is an alternative geometric proof for this inequality which is often worth keeping in mind. Suppose a circle has diameter of length $x + y$. It is clear, from the theory of

similar triangles, that the length of the perpendicular is $\sqrt{xy}$. Since the length of the perpendicular is less than or equal to the radius, we have that $\sqrt{xy} \leq \frac{x+y}{2}$, and we have equality if and only if x and y are radii.



The arithmetic-geometric mean inequality carries over to the case where we have n numbers under examination, but the method of proof is less straightforward :

Theorem 2.2 *(The Arithmetic-Geometric Mean Inequality)*
Suppose $x_1, x_2, \ldots, x_n \geq 0$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \tag{4}$$

that is, the arithmetic mean is greater than or equal to the geometric mean. The inequality is strict unless all of the numbers $x_1, x_2, \ldots, x_n$ are equal to each other.

Proof: We establish (4) by making use of the rearrangement inequality. We may suppose $x_1 \geq x_2 \geq \cdots \geq x_n$. Let

$$G = \sqrt[n]{x_1 x_2 \cdots x_n}$$

We consider the systems

$$\frac{x_1}{G}, \frac{x_1 x_2}{G^2}, \ldots, \frac{x_1 x_2 \cdots x_n}{G^n} \ (= 1)$$

and the inverses of this system, that is,

$$\frac{G}{x_1}, \frac{G^2}{x_1 x_2}, \ldots, \frac{G^n}{x_1 x_2 \cdots x_n} \ (= 1)$$

It follows from Note 1.6 that the minimal pairwise product is n. On the other hand, one of the intermediate products is

$$\frac{x_1}{G} \cdot 1 + \frac{x_1 x_2}{G^2} \cdot \frac{G}{x_1} + \frac{x_1 x_2 x_3}{G^3} \cdot \frac{G^2}{x_1 x_2} + \cdots + \frac{x_1 \cdots x_{n-1}}{G^{n-1}} \cdot \frac{G^{n-2}}{x_1 \cdots x_{n-2}} + \frac{x_1 \cdots x_n}{G^n} \cdot \frac{G^{n-1}}{x_1 \cdots x_{n-1}}$$

which simplifies to

$$\frac{x_1 + x_2 + \cdots + x_n}{G}$$

and the result follows.

That the inequality is strict unless all of the numbers $x_1, x_2, \ldots, x_n$ are equal to each other is a consequence of the fact that the rearrangement inequality is strict unless the rearrangement is trivial. ∎

Perhaps the above proof is not very enlightening. There is an alternative intuitive argument which is quite useful. Suppose we have the numbers $x_1, x_2, \ldots, x_n$ and these are arranged so that $x_1 \geq \cdots \geq x_n$. We now replace the largest number, x_1, and the smallest number, x_n, by $\frac{x_1+x_n}{2}$. We then have a new system of numbers $\frac{x_1+x_n}{2}$, $x_2, \ldots, x_{n-1}$, $\frac{x_1+x_n}{2}$. The arithmetic mean of this new system has not changed. On the other hand, since

$$\left(\frac{x_1+x_n}{2}\right)^2 \geq x_1 \cdot x_n$$

we have that the geometric mean has increased. We now repeat the process over and over : at each step the arithmetic mean is unchanged while the geometric mean increases. In the limit (a lot hides behind that innocuous phrase!) the n numbers under consideration become equal, at which time the geometric and arithmetic means become equal. It then follows that the geometric mean must have been smaller than the arithmetic mean in the first place.

Example 2.3 Show that if $a, b, c > 0$ then

$$\frac{a^3b}{c} + \frac{a^3c}{b} + \frac{b^3a}{c} + \frac{b^3c}{a} + \frac{c^3a}{b} + \frac{c^3b}{a} \geq 6abc$$

We have

$$\frac{\frac{a^3b}{c} + \frac{a^3c}{b} + \frac{b^3a}{c} + \frac{b^3c}{a} + \frac{c^3a}{b} + \frac{c^3b}{a}}{6} \geq \sqrt[6]{\frac{a^3b}{c} \cdot \frac{a^3c}{b} \cdot \frac{b^3a}{c} \cdot \frac{b^3c}{a} \cdot \frac{c^3a}{b} \cdot \frac{c^3b}{a}}$$

which simplifies to

$$\frac{a^3b}{c} + \frac{a^3c}{b} + \frac{b^3a}{c} + \frac{b^3c}{a} + \frac{c^3a}{b} + \frac{c^3b}{a} \geq 6abc$$

Example 2.4 Suppose $x, y, z > 0$. Then

$$\frac{x^3+y^3+z^3}{3} \geq \sqrt[3]{x^3y^3z^3} = xyz$$

and so

$$x^3+y^3+z^3 \geq 3xyz$$

In more generality, we have that

$$\sum_{i=1}^{n} x_i{}^n \geq n \prod_{i=1}^{n} x_i \qquad (5)$$

for $x_1, x_2, \ldots, x_n > 0$.

2.1 Exercises : the Arithmetic-Geometric Mean Inequality

1. Show that

$$a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd > 14$$

if a, b, c, and d are positive real numbers whose product is equal to 2.

2. Prove that $n! < (\frac{n+1}{2})^n$ for any integer $n > 1$.

3. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$abc \geq (a+b-c) \cdot (b+c-a) \cdot (c+a-b)$$

4. Let $a, b, c > 0$. Prove that

$$abc \geq (a+b-c) \cdot (b+c-a) \cdot (c+a-b)$$

(British MO 1981, Q3(b))

5. Given a triangle ABC, let I be the centre of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}$$

(IMO 1991, Question 1)

6. Show that of all triangles with a given perimeter, the equilateral triangle has greatest area.

7. $PQRS$ is a quadrilateral of area A. O is a point inside $PQRS$. Prove that if

$$2A = OP^2 + OQ^2 + OR^2 + OS^2$$

then $PQRS$ is a square and O its centre.

(Australian MO Inter-State final 1985, Question 3)

8. Show that if $x, y, z > 0$ and $x+y+z = 1$ then

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$$

(IMO 1984, Question 1)

9. Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geq (C_k^n)^2 a_1 a_2 \cdots a_n$$

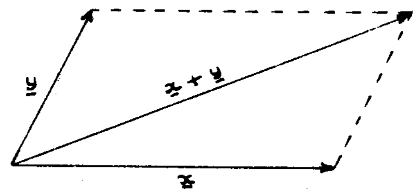
for $1 \leq k \leq n-1$.

(Asian Pacific MO 1990, Question 2)

3 The Triangle Inequality

We now consider the most elementary inequality, namely, the triangle inequality. You should already have had substantial experience with this inequality. Unfortunately, the exercises are not so easy and that's why this section was not the first section.

The easiest way to think of the triangle inequality is in the plane : suppose we have two vectors $\underline{x}$ and $\underline{y}$. Then the rules for vector addition determine the sum $\underline{x} + \underline{y}$:



The triangle inequality says that the length of $\underline{x} + \underline{y}$ is less than or equal to the length of $\underline{x}$ plus the length of $\underline{y}$, or, expressed as a formula,

$$\|\underline{x} + \underline{y}\|_2 \le \|\underline{x}\|_2 + \|\underline{y}\|_2$$

Here $\|\cdot\|_2$ is the usual symbol for the length of a vector. Recall that the length of a vector is defined to be

$$\|(x_1, x_2)\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$$

This is known as the Euclidean norm of the vector (x_1, x_2). Thus

$$\sqrt{|x_1 + y_1|^2 + |x_2 + y_2|^2} \le \sqrt{|x_1|^2 + |x_2|^2} + \sqrt{|y_1|^2 + |y_2|^2}$$

Also, when we are dealing with numbers on the real line, we have the triangle inequality

$$|x + y| \le |x| + |y|$$

for $x, y \in \mathbf{R}$.

Now these ideas are valid not only on the real line or in the plane but in any dimension. In $\mathbf{R}^n$, we define the Euclidean norm of a vector $(x_1, x_2, \ldots, x_n)$ to be

$$\|(x_1, x_2, \ldots, x_n)\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \qquad (6)$$

and then the triangle inequality will still hold. We can state this as a theorem.

Theorem 3.5 *(The Triangle Inequality)*
Suppose $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *are vectors in* $\mathbf{R}^n$. *Then*

$$\sqrt{\sum_{i=1}^{n} |x_i + y_i|^2} \le \sqrt{\sum_{i=1}^{n} |x_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i|^2} \qquad (7)$$

3.1 Exercises : the Triangle Inequality

1. In the plane there is a finite set of points, no three of which are collinear. Some points are joined to others by line segments, with each point connected to no more than one line segment. If we have a pair of intersecting line segments AB and CD we decide to replace them with AC and BD, which are opposite sides of the quadrilateral $ABCD$. In the resulting system of segments we decide to perform a similar substitution, if possible, and so on. Is it possible that such substitutions can be carried out indefinitely?

   (Tournament of the Towns 1984, Junior Question 3)

2. Let f and g be real-valued functions defined for all real values of x and y and satisfying the equation

   $$f(x + y) + f(x - y) = 2f(x)\, g(y)$$

   for all $x, y \in \mathbf{R}$. Prove that if $f(x)$ is not identically 0, and if $|f(x)| \le 1$ for all x, then $|g(y)| \le 1$ for all y.

   (IMO 1972, Question 5)

3. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with $n > 3$ vertices, and let p be its perimeter. Prove that

   $$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2 \, .$$

   where $[x]$ denotes the greatest integer not exceeding x.

   (IMO 1984, Question 5)

4 The Cauchy-Schwartz Inequality

Theorem 4.1 *(The Cauchy-Schwartz Inequality)*
Suppose $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *are vectors in* $\mathbf{R}^n$. *Then*

$$\sum_{i=1}^{n} x_i y_i \le \sqrt{\sum_{i=1}^{n} |x_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |y_i|^2} \qquad (8)$$

This inequality is strict unless one of the two vectors $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *is a multiple of the other.*

Proof: We will establish this inequality by making use of the theory of discriminants of quadratics. Suppose we have two vectors $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbf{R}^n$. We consider

$$f(t) = \sum_{i=1}^{n} (x_i t + y_i)^2$$

which is a function of the real variable t. It is clear that this function takes on positive values for all $t \in \mathbf{R}$. Now multiplying out and grouping like terms we get

$$f(t) = \left(\sum_{i=1}^{n} x_i{}^2\right) t^2 + \left(2\sum_{i=1}^{n} x_i y_i\right) t + \left(\sum_{i=1}^{n} y_i{}^2\right)$$

which is a quadratic in the variable t. So f is a quadratic which takes on only positive values, and so it must have discriminant $\Delta \leq 0$. That means

$$\left(2\sum_{i=1}^{n} x_i y_i\right)^2 - 4 \cdot \left(\sum_{i=1}^{n} x_i{}^2\right) \cdot \left(\sum_{i=1}^{n} y_i{}^2\right) \leq 0$$

which simplifies to

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i{}^2\right) \cdot \left(\sum_{i=1}^{n} y_i{}^2\right)$$

which is what is required. We can have equality if and only if $\Delta = 0$, which occurs if and only if the function $f(t)$ takes on the value 0 for some (in fact, exactly one) value of t. That would mean that

$$\sum_{i=1}^{n} (x_i t + y_i)^2 = 0$$

which means that $x_i t + y_i = 0$ for $i = 1, 2, \ldots, n$. That means that one of the two vectors $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ is a multiple of the other. ■

The test for when the Cauchy-Schwartz inequality is an equality is crucial in the solution of a number of problems, so it is important that it is properly understood. For example, the vectors $(3, 4, 5)$ and $(6, 8, 10)$ are multiples of each other, since $(6, 8, 10) = 2 \cdot (3, 4, 5)$. And then we get

$$3 \cdot 6 + 4 \cdot 8 + 5 \cdot 10 = 100 = \sqrt{3^2 + 4^2 + 5^2} \cdot \sqrt{6^2 + 8^2 + 10^2}$$

However, 'is a multiple of' is not a symmetric relation : $(0, 0, 0)$ is a multiple of $(3, 4, 5)$ but $(3, 4, 5)$ is not a multiple of $(0, 0, 0)$. It is this rather annoying little detail that made us say that mouthful in the statement of the theorem, rather than just say : 'This inequality is strict unless $(x_1, x_2, \ldots, x_n)$ is a multiple of $(y_1, y_2, \ldots, y_n)$', which would be a false statement.

The Cauchy-Schwartz inequality has already been stated in two equivalent forms, and now we state a few more which are obvious but deserve to be made familiar. Firstly we have

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i{}^2 \cdot \sum_{i=1}^{n} y_i{}^2 \qquad (9)$$

The following two forms hold only if $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are known to be positive:

$$\sum_{i=1}^{n} \sqrt{x_i y_i} \leq \sqrt{\sum_{i=1}^{n} x_i} \cdot \sqrt{\sum_{i=1}^{n} y_i} \qquad (10)$$

$$\left(\sum_{i=1}^{n} \sqrt{x_i y_i}\right)^2 \leq \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i \qquad (11)$$

and the test for when there is equality remains the same. This last form is especially important because it is never obvious when it is begging to be used! In the final analysis: the Cauchy-Schwartz inequality gives important information about the product of two sums. Whenever information is needed about such a product in a problem, use of the Cauchy-Schwartz inequality should be suggested.

Note 4.2 It follows from the rearrangement inequality that in dealing with the case of the Cauchy-Schwartz inequality where all of the numbers are positive we may whenever it is convenient assume $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. In (8) this assumption maximises the left hand side and leaves the right hand side unchanged.

Example 4.3 Use the Cauchy-Schwartz inequality to find the maximum value of

$$f(x, y, z) = 3x + y + 2z$$

given that $x^2 + y^2 + z^2 = 1$.

We consider the two vectors (x, y, z) and $(3, 1, 2)$. Then

$$(3x + y + 2z)^2 \leq \left(x^2 + y^2 + z^2\right) \cdot \left(3^2 + 1^2 + 2^2\right) = 14$$

and so $3x + y + 2z \leq \sqrt{14}$. It would be reasonable to hypothesise that the maximum value is in fact $\sqrt{14}$, but this would be jumping the gun. So far we have only shown that $\sqrt{14}$ is an upper bound for $3x + y + 2z$, not the least upper bound.

To do this, we test for when the Cauchy-Schwartz inequality yields an equality : we need some $\lambda \in \mathbf{R}$ such that $x = 3\lambda,\ y = \lambda,\ z = 2\lambda$. It is then easy to show that $\lambda = \frac{1}{\sqrt{14}}$ and hence

$$x = \frac{3}{\sqrt{14}}, \quad y = \frac{1}{\sqrt{14}}, \quad z = \frac{2}{\sqrt{14}}$$

gives us values for which the function $3x + y + 2z$ does in fact take on the value $\sqrt{14}$.

4.1 Exercises : the Cauchy-Schwartz Inequality

1. Show that if $x_1, x_2, \ldots, x_n$ are all positive then

$$\left(\sum_{i=1}^{n} x_i\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \geq n^2 \qquad (12)$$

and determine when equality occurs.

2. Show that

$$n \cdot \sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2 \tag{13}$$

3. Use the Cauchy-Schwartz inequality to find the minimum value of

$$f(x,y,z) = x^2 + y^2 + \frac{z^2}{2}$$

given that $x + y + z = 10$.

4. Suppose $x, y, z > 0$. Use the Cauchy-Schwartz inequality to show that

$$(x+y+z)\left[\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right] \geq \frac{9}{2}$$

and hence show that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$$

5. Show that if $a + b + c = 1$ then $a^2 + b^2 + c^2 \geq \frac{1}{3}$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$.

6. Prove that the polynomial

$$x^4 + x^3 + ax^2 + bx + c$$

does not have all its roots real when $a, b, c \in \mathbf{R}$ and $a > \frac{3}{8}$.

7. Prove that the polynomial

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

does not have all its roots real when $a, b, c, d, e \in \mathbf{R}$ and $2a^2 < 5b$.
(USA MO, 1982, Question 2)

8. The numbers $a, b, a_2, a_3, \ldots, a_{n-2}$ are all real, and $ab \neq 0$. All the roots of the equation

$$ax^n - ax^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 - n^2bx + b = 0$$

are real and positive. Prove that all the roots are mutually equal.
(Australian MO 1986, Question 6)

9. Given that a, b, c, d, e are real numbers such that

$$\begin{aligned} a+b+c+d+e &= 8 \\ a^2+b^2+c^2+d^2+e^2 &= 16 \end{aligned}$$

determine the maximum value of e.

10. Suppose a, b, c, d, m, n are positive integers satisfying

$$\begin{aligned} a^2+b^2+c^2+d^2 &= 1989 \\ a+b+c+d &= m^2 \\ \max\{a,b,c,d\} &= n^2 \end{aligned}$$

Determine a, b, c, d, m, n.
(Submitted by Ireland to the IMO, 1989)

11. Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geq (C_k^n)^2 a_1 a_2 \cdots a_n$$

for $1 \leq k \leq n-1$.
(Asian Pacific MO 1990, Question 2)

12. Suppose that $a_1, a_2, \ldots, a_n$ are real $(n > 1)$ and

$$A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2$$

Prove that $A < 2a_ia_j$ for $1 \leq i < j \leq n$.
(Putnam Competition No 38, Question B5)

13. Find all real numbers A for which there exist non-negative real numbers $x_1, \ldots, x_5$ satisfying the relations

$$\sum_{k=1}^{5} kx_k = A \quad ; \quad \sum_{k=1}^{5} k^3x_k = A^2 \quad ; \quad \sum_{k=1}^{5} k^5x_k = A^3$$

(IMO 1979, Question 5)

14. P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.
(IMO 1981, Question 1)

15. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, xz-plane, xy-plane respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$$

where $|A|$ denotes the number of elements in the finite set A.

Note : the orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.

(IMO 1992, Question 5)

5 Jensen's Inequality

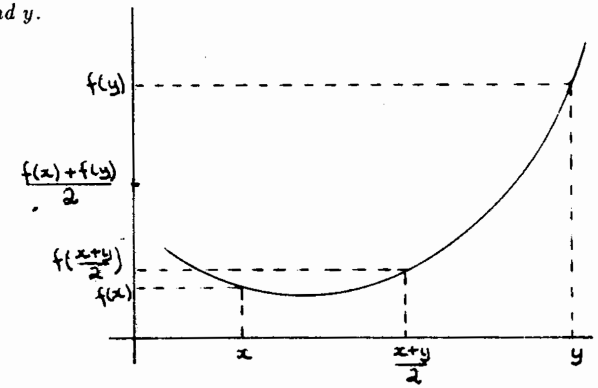
Is this section we discuss an inequality which is quite different to the inequalities we have discussed in previous sections : all of our previous inequalities involve vectors from $\mathbf{R}^n$, or more generally, finite sequences of numbers, whereas Jensen's inequality is concerned with a special class of functions on the real line known as convex or concave functions. These functions derive their names from the corresponding terms in the study of optics.

Throughout we suppose that a function f is defined on some interval of the real line. Typically this will be the set $[0,\infty)$ of all non-negative numbers.

Definition 5.1 *f is said to be convex if*

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$$

for all x and y.



Examples 5.2 (a) The functions $f(x)=x^2$ and $f(x)=x^4$, defined on all of $\mathbf{R}$, are convex.

(b) The functions $f(x)=x^n$ for any $n \in \mathbf{N}$, defined on the interval $[0,\infty)$, are all convex.

(c) Any quadratic function with positive leading coefficient is convex.

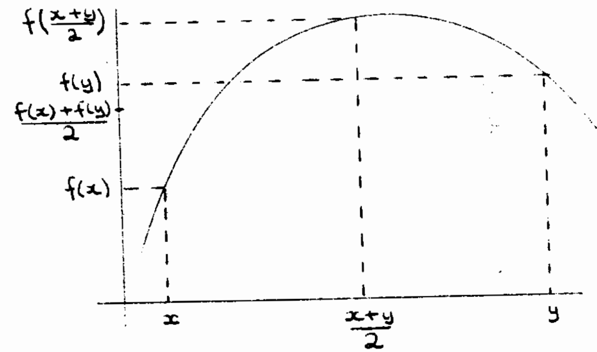
(c) The function $f(x)=\cos x$, defined on the interval $[90°, 270°]$, is convex.

(d) The exponential functions are convex.

Definition 5.3 *f is said to be concave if*

$$\frac{f(x)+f(y)}{2} \leq f\left(\frac{x+y}{2}\right)$$

for all x and y.



Examples 5.4 (a) The functions $f(x)=-x^2$, defined on all of $\mathbf{R}$, is concave. In fact, if f is a convex function, then $-f$ is concave, and conversely.

(b) The function $f(x)=x^3$, defined on all of $\mathbf{R}$, is concave in the interval $(-\infty,0]$ and convex in the interval $[0,\infty)$.

(c) The function $f(x)=\sqrt{x}$ is concave on its natural domain, $[0,\infty)$.

(d) The logarithmic functions are concave.

(e) The function $f(x)=\sin x$, defined on the interval $[0°, 180°]$, is concave.

(f) The function $f(x)=\cos x$, defined on the interval $[-90°, 90°]$, is concave.

We have not stopped to prove that these functions are convex or concave; very often this is quite difficult. We can, for example, verify that the function $f(x)=\sin x$ is concave on $[0°, 180°]$:

$$\frac{\sin x+\sin y}{2} = \sin\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right) \leq \sin\left(\frac{x+y}{2}\right)$$

For those of you who know anything about calculus : if a function has a second derivative, then the function is convex if and only if its second derivative is positive and concave if and only if its second derivative is negative. This provides a quick test for convexity and concavity. (In this regard we should note that in some calculus textbooks convexity is called 'concave up' and concavity is called 'concave down'.)

For our purposes you should draw a diagram of each of the functions listed above and satisfy yourself that they seem to be of the type mentioned.

Theorem 5.5 *(Jensen's inequality)*

(a) Suppose f is convex. Then for any $x_1, x_2, \ldots, x_n$ belonging to the definition interval we have

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+\cdots+f(x_n)}{n}$$

(b) Suppose f is concave. Then for any $x_1, x_2, \ldots, x_n$ belonging to the definition interval we have

$$\frac{f(x_1)+\cdots+f(x_n)}{n} \leq f\left(\frac{x_1+\cdots+x_n}{n}\right)$$

Provided that the function f is not a straight line on any part of its domain, the inequality is strict : there will be equality if and only if the numbers $x_1, \cdots, x_n$ are all equal.

Proof: We establish the case where f is convex; by judicious insertion of minus signs this will prove the concave case too.

We will establish this result by means of what is often referred to as 'backwards induction'. That the result holds for $n = 1$ is a tautology and that it holds for $n = 2$ is the actual definition of convexity. Suppose the result holds for some value of n, we show it holds for $2n$. We have

$$\begin{aligned} f\left(\frac{x_1+\cdots+x_{2n}}{2n}\right) &= f\left(\frac{\frac{x_1+\cdots+x_n}{n}+\frac{x_{n+1}+\cdots+x_{2n}}{n}}{2}\right) \\ &\leq \frac{f\left(\frac{x_1+\cdots+x_n}{n}\right)+f\left(\frac{x_{n+1}+\cdots+x_{2n}}{n}\right)}{2} \\ &\leq \frac{\frac{f(x_1)+\cdots+f(x_n)}{n}+\frac{f(x_{n+1})+\cdots+f(x_{2n})}{n}}{2} \\ &= \frac{f(x_1)+\cdots+f(x_{2n})}{2n} \end{aligned}$$

Thus the result holds true for all positive powers of 2. To finish the proof, we need to show the 'backwards' step : if the result is true for $n+1$ then it is true for n. Suppose

we have the values $x_1, \ldots, x_n$. Let $x_{n+1} = \frac{x_1+\cdots+x_n}{n}$. Then note that

$$\frac{x_1+\cdots+x_n}{n} = \frac{x_1+\cdots+x_{n+1}}{n+1}$$

and so

$$\begin{aligned} f\left(\frac{x_1+\cdots+x_n}{n}\right) &= f\left(\frac{x_1+\cdots+x_{n+1}}{n+1}\right) \\ &\leq \frac{f(x_1)+\cdots+f(x_{n+1})}{n+1} \\ &= \frac{f(x_1)+\cdots+f(x_n)}{n+1} + \frac{f\left(\frac{x_1+\cdots+x_n}{n}\right)}{n+1} \\ \Rightarrow (n+1)\cdot f\left(\frac{x_1+\cdots+x_n}{n}\right) &\leq f(x_1)+\cdots+\mathrm{Ch}(x_n)+f\left(\frac{x_1+\cdots+x_n}{n}\right) \\ \Rightarrow f\left(\frac{x_1+\cdots+x_n}{n}\right) &\leq \frac{f(x_1)+\cdots+f(x_n)}{n} \end{aligned}$$

■

Example 5.6 Show that if $x, y \geq 0$ then

$$x^3+y^3 \geq \frac{1}{4}(x+y)^3$$

The cubic function $f(x) = x^3$ is convex for $x \geq 0$. Hence by Jensen's inequality we have

$$\frac{x^3+y^3}{2} \geq \left(\frac{x+y}{2}\right)^3$$

from which the result follows by multiplying through by 2.

Example 5.7 The square root function is concave on $[0, \infty)$. For any $x_1, \ldots, x_n > 0$ we have

$$\frac{\sqrt{x_1}+\cdots+\sqrt{x_n}}{n} \leq \sqrt{\frac{x_1+\cdots+x_n}{n}}$$

and so

$$(\sqrt{x_1}+\cdots+\sqrt{x_n})^2 \leq n\cdot(x_1+\cdots+x_n)$$

5.1 Exercises : Jensen's Inequality

1. Show that if $x, y, z \geq 0$ and $x+y+z = 1$ then

   $$x^2+y^2+z^2 \geq \frac{1}{3}$$

   and hence show that

   $$xy+yz+zx \leq \frac{1}{3}$$

2. Draw graphs for each of the six trigonometric functions on $(0°, 180°)$ and decide on concavity and convexity of these functions.

3. Suppose a, b, c are the angles of any triangle. Show that

$$\begin{aligned} \sin a + \sin b + \sin c &\le \frac{3\sqrt{3}}{2} \\ \csc a + \csc b + \csc c &\ge 2\sqrt{3} \\ \cos a + \cos b + \cos c &\le \frac{3}{2} \\ \cot a + \cot b + \cot c &\ge \sqrt{3} \end{aligned}$$

and if the triangle is acute-angled, then

$$\begin{aligned} \tan a + \tan b + \tan c &\ge 3\sqrt{3} \\ \sec a + \sec b + \sec c &\ge 6 \end{aligned}$$

Show that no suitable inequality holds in general for the tangent and secant functions, that is, the functions $\tan a + \tan b + \tan c$ and $\sec a + \sec b + \sec c$ are unbounded from below.

4. Prove that amongst all triangles whose incircle has radius equal to 1, the equilateral triangle has the shortest perimeter.

(Nordic Mathematical Contest, 1992)

5. Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $< PAB, < PBC, < PCA$ is less than or equal to 30 degrees.

(IMO 1991, Question 5)

6 Solutions

6.1 The Rearrangement Inequality

1. Without loss of generality $a \ge b \ge c$. Then $a + b \ge a + c \ge b + c$ and so $\frac{1}{b+c} \ge \frac{1}{a+c} \ge \frac{1}{a+b}$. Therefore

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\ge \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b} \\ \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\ge \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b} \end{aligned}$$

and so

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \ge \frac{b+c}{b+c} + \frac{c+a}{c+a} + \frac{a+b}{a+b} = 3$$

and of course we have equality in the case of an equilateral triangle.

For the other inequality, we perform the Ravi substitution. Then

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= \frac{x+y}{x+2z+y} + \frac{y+z}{y+2x+z} + \frac{z+x}{z+2y+x} \\ &< \frac{x+y}{x+z+y} + \frac{y+z}{y+x+z} + \frac{z+x}{z+y+x} \\ &= 2 \end{aligned}$$

2. After applying the Ravi substitution we find that we are required to show that

$$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y \ge 6xyz$$

There are at least three ways of establishing this inequality:-

(a) The expression is symmetric in x, y, z and so we may suppose $x \ge y \ge z$. Then

$$xy \ge xz \ge yz$$

and hence

$$x \cdot xy + y \cdot yz + z \cdot xz \ge x \cdot yz + y \cdot xz + z \cdot xy$$

and likewise

$$x \cdot xz + y \cdot xy + z \cdot yz \ge x \cdot yz + y \cdot xz + z \cdot xy$$

both from the rearrangement inequality. Adding these two equations together we get the required result.

(b) The second method uses (2) :-

$$\begin{aligned} & x^2y + x^2z + y^2x + y^2z + z^2x + z^2y \\ &= \left[\left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{x}{z} + \frac{z}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right)\right] \cdot xyz \\ &\ge [2 + 2 + 2] \cdot xyz \\ &= 6xyz \end{aligned}$$

(c) Finally, the Arithmetic-Geometric mean inequality can be used : try this question again when you get there.

3. Without loss of generality, $a \geq b \geq c$. Then $ab \geq ac \geq bc$ and $\frac{1}{bc} \geq \frac{1}{ac} \geq \frac{1}{ab}$. Hence

$$\begin{aligned} \frac{a^8+b^8+c^8}{a^3b^3c^3} &= \frac{a^5}{b^3c^3}+\frac{b^5}{c^3a^3}+\frac{c^5}{a^3b^3} \\ &\geq \frac{a^5}{a^3c^3}+\frac{b^5}{a^3b^3}+\frac{c^5}{b^3c^3} \\ &= \frac{a^2}{c^3}+\frac{b^2}{a^3}+\frac{c^2}{b^3} \\ &\geq \frac{a^2}{a^3}+\frac{b^2}{b^3}+\frac{c^2}{c^3} \\ &= \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \end{aligned}$$

4. Without loss of generality $a \geq b \geq c$. Then $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{c}$ and $a^2 \geq b^2 \geq c^2$. Hence

$$\left(\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}\right)+\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right) \geq 2(a+b+c)$$

and

$$2\left(\frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}\right) \geq \left(\frac{a^3}{ca}+\frac{b^3}{ab}+\frac{c^3}{bc}\right)+\left(\frac{a^3}{ba}+\frac{b^3}{bc}+\frac{c^3}{ac}\right)$$

5. After making the Ravi substitution and some lengthy but routine manipulation, we see that the required inequality reduces to showing that

$$x^3z+y^3x+z^3y \geq xyz(x+y+z)$$

Now clearly this expression is not symmetric in the variables x, y, z, so we cannot, for example, simply suppose $x \geq y \geq z$. Thus in principle there are six orderings of the variables x, y, z to consider. But always there will be some cyclic symmetry as is evident here:-

First suppose $x \geq y \geq z$. Then $xy \geq xz \geq yz$ and $x^2 \geq y^2 \geq z^2$. Hence, using these two systems in the rearrangement inequality, we get that

$$xy \cdot y^2 + xz \cdot x^2 + yz \cdot z^2 \geq xy \cdot z^2 + xz \cdot y^2 + yz \cdot x^2$$

By cyclic symmetry this also deals with the case where $y \geq z \geq x$ and $z \geq x \geq y$.

A similar argument works for the case $x \geq z \geq y$ and this also covers the case $y \geq x \geq z$ and $z \geq y \geq x$.

6. Whatever order of size x, y, z are in, x^2, y^2, z^2 are in the same order. Hence, by Note 1.6 we have

$$x \cdot x^2 + y \cdot y^2 + z \cdot z^2 \geq x \cdot z^2 + y \cdot x^2 + z \cdot y^2$$

7. Suppose $n \in \mathbf{N}$. By the rearrangement inequality the expression $\sum_{k=1}^{n} \frac{a_k}{k^2}$ will be smallest when $a_n > \cdots > a_2 > a_1$, so we can suppose this is the case. Then in fact $a_k \geq k$ for $1 \leq k \leq n$. Thus

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{k}{k^2} = \sum_{k=1}^{n} \frac{1}{k}$$

8. From the rearrangement inequality we have that

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\sigma(i)}$$

for any permutation σ. We take the following n permutations in turn:

$$\begin{array}{llllll} 1 & 2\ 3 & \cdots & n-2 & n-1 & n; \\ 2 & 3\ 4 & \cdots & n-1 & n & 1; \\ 3 & 4\ 5 & \cdots & n & 1 & 2; \\ \cdots; & & & & & \\ n & 1\ 2 & \cdots & n-3 & n-2 & n-1. \end{array}$$

Adding up the n resulting equations up gives us the required result.

9. Since $x, y > 1$ we have that $1 < x \leq x^2 \leq \cdots \leq x^{n-1}$ and $1 < y \leq y^2 \leq \cdots \leq y^{n-1}$. Hence by the Chebyshev inequality we have

$$n \cdot \sum_{i=0}^{n-1} x^i y^i \leq \sum_{i=0}^{n-1} x^i \cdot \sum_{i=0}^{n-1} y^i$$

which, when using known facts about geometric progressions, simplifies to the required expression.

6.2 The Arithmetic-Geometric Mean Inequality

1. By the Arithmetic-geometric mean inequality we have

$$\begin{aligned} \frac{ab+ac+ad+bc+bd+cd}{6} &\geq \sqrt[6]{ab \cdot ac \cdot ad \cdot bc \cdot bd \cdot cd} \\ &= \sqrt{abcd} \\ &= \sqrt{2} \end{aligned}$$

and likewise

$$\begin{aligned}\frac{a^2+b^2+c^2+d^2}{4} &\geq \sqrt[4]{a^2\cdot b^2\cdot c^2\cdot d^2}\\ &= \sqrt{abcd}\\ &= \sqrt{2}\,.\end{aligned}$$

Thus

$$a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd \geq 6\sqrt{2}+4\sqrt{2} > 14$$

2. Consider the numbers $1,2,3,\ldots,n$. By the Arithmetic-Geometric mean inequality,

$$\sqrt[n]{1\cdot 2\cdot 3\cdots n} \leq \frac{1+2+\cdots+n}{n}$$

Simplifying this gives us that $n! \leq (\frac{n+1}{2})^n$. But certainly this inequality is strict because the numbers $1,2,\ldots,n$ are not equal - because $n>1$.

3. After applying the Ravi substitution we find that we are required to show that

$$x^2y+x^2z+y^2x+y^2z+z^2x+z^2y \geq 6xyz$$

which is easy to establish :

$$\begin{aligned}\frac{x^2y+x^2z+y^2x+y^2z+z^2x+z^2y}{6} &\geq \sqrt[6]{x^2y\cdot x^2z\cdot y^2x\cdot y^2z\cdot z^2x\cdot z^2y}\\ &= xyz\end{aligned}$$

4. We may suppose $a \geq b \geq c > 0$. Now these numbers form the lengths of the sides of a triangle iff $a < b+c$. If this is the case, then the result follows from the previous exercise. If it isn't, then the right hand side is negative, and the left hand side is positive, and so the inequality is quite trivial.

5. Now $\frac{AI}{IA'} = \frac{c}{x}$ since BI bisects $< B$. Hence

$$\frac{AI}{IA'} = \frac{b+c}{x+y} = \frac{b+c}{a}$$

which is a standard result that deserves to be memorised. It is now easy to see that we are simply required to prove that

$$\frac{1}{4} < \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \leq \frac{8}{27}$$

From the Arithmetic-Geometric mean inequality we have that

$$\sqrt[3]{\frac{a+b}{a+b+c}\cdot\frac{b+c}{a+b+c}\cdot\frac{c+a}{a+b+c}} \leq \frac{1}{3}\left[\frac{a+b}{a+b+c}+\frac{b+c}{a+b+c}+\frac{c+a}{a+b+c}\right] = \frac{2}{3}$$

and so

$$\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \leq \frac{8}{27}$$

For the other inequality, we use the Ravi substitution, and from this it easily follows that we must show

$$(y+2x+z)(z+2y+x)(x+2z+y) > 2(x+y+z)^3$$

Now this is apparently true, since we can consider the following factorisation of the left hand side :-

$$\begin{aligned}&(y+2x+z)(z+2y+x)(x+2z+y)\\ =\ &[(y+x+z)+x]\cdot[(z+y+x)+y]\cdot[(x+z+y)+z]\\ =\ &(y+x+z)^3\\ +\ &x\cdot(z+y+x)^2+(y+x+z)\cdot y\cdot(x+z+y)+(y+x+z)^2\cdot z\\ +\ &x\cdot y\cdot(x+z+y)+(y+x+z)\cdot y\cdot z+x\cdot(z+y+x)\cdot z\\ +\ &xyz\\ >\ &2(x+y+z)^3\end{aligned}$$

6. By Heron's formula for the area of a triangle, we have

$$A = \sqrt{xyz(x+y+z)} \tag{14}$$

where $a=x+y,\ b=y+z,\ c=z+x$ are the lengths of the sides of the triangle. By the Arithmetic-Geometric mean inequality,

$$xyz \leq \left(\frac{x+y+z}{3}\right)^3$$

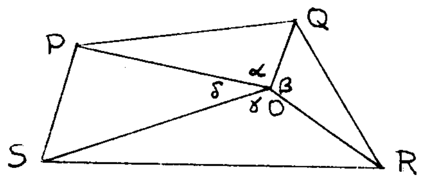
and so

$$\begin{aligned}A &= \sqrt{xyz(x+y+z)}\\ &\leq \sqrt{\left(\frac{x+y+z}{3}\right)^3(x+y+z)}\\ &= \frac{(x+y+z)^2}{3\sqrt{3}}\end{aligned}$$

And this maximum is attained when the Arithmetic-Geometric mean inequality is an equality i.e. when the terms used there are all equal, i.e. when $x=y=z$. This happens when the triangle is equilateral.

7. With angles at O as indicated,

$$\begin{aligned}4A &= 2(OP\cdot OQ\cdot\sin\alpha+\cdots+\cdots+\cdots)\\ &\leq (OP^2+OQ^2)\sin\alpha+\cdots+\cdots+\cdots\\ &\leq (OP^2+OQ^2)+\cdots+\cdots+\cdots\\ &= 2(PO^2+OQ^2+OR^2+OS^2)\end{aligned}$$



The first inequality is strict unless $OP = OQ = OR = OS$ and the second is strict unless $\sin\alpha = \cdots = 1$. Hence equality implies that the diagonals are of equal length and are bisected at right angles at O. So the quadrilateral is a square with O as its centre.

8. From the hypotheses it is clear that at least one of the numbers, say z, is less than or equal to $\frac{1}{2}$. Thus

$$xy + yz + zx - 2xyz = (x+y)z + (1-2z)xy \geq 0$$

Using the same trick and applying the Arithmetic-Geometric mean inequality,

$$\begin{aligned} & xy + yz + zx - 2xyz - \frac{7}{27} \\ = \ & (x+y)z + (1-2z)xy - \frac{7}{27} \\ = \ & (1-z)z + (1-2z)xy - \frac{7}{27} \\ \leq \ & (1-z)z + (1-2z)\left(\frac{x+y}{2}\right)^2 - \frac{7}{27} \\ = \ & (1-z)z + (1-2z)\left(\frac{1-z}{2}\right)^2 - \frac{7}{27} \\ = \ & \frac{1}{4\cdot 27}\cdot[4\cdot 27\cdot(1-z)z + 27\cdot(1-2z)(1-z)^2 - 7\cdot 4] \\ = \ & \frac{1}{4\cdot 27}\cdot[-1 + 27z^2 - 54z^3] \\ = \ & -\frac{1}{4\cdot 27}(3z-1)^2(6z+1) \\ \leq \ & 0 \end{aligned}$$

9. Consider a set of k of the numbers $a_1, a_2, \ldots, a_n$, denoted by $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$. Then

$$S_k = \sum_{1\leq i_1<\cdots<i_k\leq n} a_{i_1}a_{i_2}\cdots a_{i_k}$$

There are C_k^n objects in this sum, and so by the arithmetic-geometric mean inequality we have

$$S_k \ \geq \ C_k^n \left(\prod_{1\leq i_1<\cdots<i_k\leq n} a_{i_1}a_{i_2}\cdots a_{i_k}\right)^{\frac{1}{C_k^n}}$$

$$\begin{aligned} &= \ C_k^n \left((a_1a_2\cdots a_n)^{C_{k-1}^{n-1}}\right)^{\frac{1}{C_k^n}} \\ &= \ C_k^n\,(a_1a_2\cdots a_n)^{\frac{k}{n}} \end{aligned}$$

since each a_{i_j} appears C_{k-1}^{n-1} times in total.

Hence

$$S_kS_{n-k} \geq C_k^nC_{n-k}^n(a_1a_2\cdots a_n)^{\frac{k}{n}+\frac{n-k}{n}} = (C_k^n)^2a_1a_2\cdots a_n$$

(You would have noticed that this solution provides more than what the question asked for. For a simpler minimalistic solution, see the section on the Cauchy-Schwartz inequality.)

6.3 The Triangle Inequality

1. Since there are finitely many points, there are only finitely many possible configurations for the line segments. Thus the only way we could carry out this process indefinitely is if we could find some repeating 'loop'.

   However, this is impossible : suppose AC intersects BD at E, and we replace them with AB and CD. By the triangle inequality

$$AC + BD = AE + EB + CE + ED > AB + CD$$

   Hence the total lengths of all the segments decreases with each substitution, and so we cannot 'loop'.

2. Let M be the supremum of all the values of $|f(x)|$. Note that we are told that $M > 0$. Then for any $x, y \in \mathbf{R}$

$$\begin{aligned} 2|f(x)|\,|g(y)| &= |f(x+y) + f(x-y)| \\ &\leq |f(x+y)| + |f(x-y)| \\ &\leq 2M \end{aligned}$$

   and so $|g(y)| \leq \frac{M}{|f(x)|}$ for any x such that $f(x) \neq 0$. We now choose $x \in \mathbf{R}$ such that $|f(x)|$ is very (one says arbitrarily) close to M. It then follows that $|g(y)| \leq 1$.

3. Consider the convex polygon $A_0A_1A_2\ldots A_{n-1}$, with the subscripts reduced modulo n. Let A_iA_j be a diagonal. Then by the triangle inequality,

$$A_iA_j + A_{i+1}A_{j+1} > A_iA_{i+1} + A_jA_{j+1}$$

   When we sum these inequalities over all $\frac{1}{2}n(n-3)$ diagonals A_iA_j, each diagonal occurs twice on the left, and each side occurs $n-3$ times on the right; so we get $2d > (n-3)p$, or $n - 3 < \frac{2d}{p}$.

To obtain the upper bound, consider the diagonal A_iA_j. Since it is the shortest path connecting A_i and A_j, it is shorter than each polygonal path joining its endpoints. Thus we have both

$$\begin{aligned} A_iA_j &< A_iA_{i+1} + \cdots + A_{j-1}A_j \\ A_iA_j &< A_jA_{j+1} + \cdots + A_{i-1}A_i \end{aligned}$$

If n is odd, say $n = 2k-1$, use for any diagonal A_iA_j that of the two inequalities above with fewer terms on the right. When we sum these $\frac{1}{2}n(n-3)$ inequalities, we get a d on the left, and on the right a sum of side lengths where each side appears exactly $2+3+\cdots+k-1 = \frac{1}{2}k(k-1)-1$ times. Therefore

$$d < \frac{p}{2}(k(k-1)-2) = \frac{p}{2}\left(\frac{n+1}{2}\cdot\frac{n-1}{2} - 2\right)$$

If n is even, say $n = 2k$, use the same inequalities as above except for diagonals A_iA_{i+k} : for these 'diameters' use the inequalities $A_iA_{i+k} \le \frac{p}{2}$. Summing these $\frac{1}{2}n(n-3)$ inequalities, we get

$$\begin{aligned} d &< k\cdot\frac{p}{2} + \frac{p}{2}(k(k-1)-2) \\ &= \frac{p}{2}(k(k-1)-2+k) \\ &= \frac{p}{2}\left(k^2-2\right) \\ &= \frac{p}{2}\left(\frac{n^2}{4}-2\right) \end{aligned}$$

Finally it is very easy to show that for even n, $\frac{n^2}{4} = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ and for odd n, $\frac{n^2-1}{4} = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$.

6.4 The Cauchy-Schwartz Inequality

1. By (11) we have

$$\left(\sum_{i=1}^{n} x_i\right)\cdot\left(\sum_{i=1}^{n}\frac{1}{x_i}\right) \ge \left(\sum_{i=1}^{n}\sqrt{x_i\frac{1}{x_i}}\right)^2 = n^2$$

Clearly there can be equality if and only if the x_i's are all equal.

2. This follows from (9) by putting all the y_i's $= 1$. (It also follows from the Chebyshev inequality - see Section 1.)

3. We consider the two vectors $\left(x, y, \frac{z}{\sqrt{2}}\right)$ and $(1, 1, \sqrt{2})$. Then

$$10 = x+y+z \le \left(x^2+y^2+\frac{1}{2}z^2\right)\cdot(1+1+2) = 4\left(x^2+y^2+\frac{1}{2}z^2\right)$$

and so the minimum value is at most 2.5. As usual, we test for when the Cauchy-Schwartz inequality yields an equality : we need some $\lambda \in \mathbf{R}$ such that

$$x = \lambda, \quad y = \lambda, \quad \frac{z}{\sqrt{2}} = \sqrt{2}\lambda$$

It is then easy to show that $\lambda = 2.5$ and hence

$$x = 2.5, \quad y = 2.5, \quad z = 5$$

gives us values for which the function does in fact take on the value 2.5.

4. Making the substitution $P = x+y, Q = y+z, R = z+x$ we find that we are required to show that

$$(P+Q+R)\left[\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}\right] \ge 9$$

which follows from (12) in the case $n = 3$.

The second part follows immediately from the first.

5. By considering the vectors (a, b, c) and $(1, 1, 1)$ we have that

$$1 = (a+b+c)^2 \le (a^2+b^2+c^2)\cdot 3$$

and the first result follows. The other is immediate from (12).

6. Let the roots of the polynomial be $r_1, \ldots, r_4$. Then

$$\sum_i r_i = -1 \quad , \quad \sum_{i\ne j} r_ir_j = a > \frac{3}{8}$$

Hence

$$\begin{aligned} 1 &= (-1)^2 \\ &= \left(\sum_i r_i\right)^2 \\ &= 2\cdot\sum_{i\ne j} r_ir_j + \sum_i r_i^2 \\ &= 2a + \sum_i r_i^2 \\ &> \frac{6}{8} + \sum_i r_i^2 \\ \Rightarrow \frac{1}{4} &> \sum_i r_i^2 \end{aligned}$$

That this is impossible follows from (13).

7. Let the roots of the polynomial be $r_1, \ldots, r_5$. Then $\sum_i r_i = -a$ and $\sum_{i \neq j} r_i r_j = b$. Hence

$$\begin{aligned} a^2 &= \left(\sum_i r_i\right)^2 \\ &= 2 \cdot \sum_{i \neq j} r_i r_j + \sum_i r_i^2 \\ &= 2b + \sum_i r_i^2 \\ &> \frac{4a^2}{5} + \sum_i r_i^2 \\ \Rightarrow \frac{a^2}{5} &> \sum_i r_i^2 \end{aligned}$$

The result now follows as in the previous exercise.

8. We have, by the usual connections between roots and coefficients of polynomials, that if $x_1, x_2, \ldots, x_n$ are the roots then

$$\begin{aligned} x_1 + \cdots + x_n &= 1 \\ \frac{1}{x_1} + \cdots + \frac{1}{x_n} &= n^2 \end{aligned}$$

The result now follows from (12).

9. We put

$$\begin{aligned} 8 - e &= a + b + c + d \\ 16 - e^2 &= a^2 + b^2 + c^2 + d^2 \end{aligned}$$

By the Cauchy-Schwartz inequality we have

$$a + b + c + d \leq \sqrt{1+1+1+1}\sqrt{a^2 + b^2 + c^2 + d^2}$$

and so $(8-e)^2 \leq 4(16 - e^2)$. Upon simplifying we find that $e(5e - 16) \leq 0$. It follows that the maximum value of e is $\frac{16}{5}$, and this is indeed attained when $a = b = c = d = \frac{16}{5}$.

10. We may suppose that $n^2 = a \geq b \geq c \geq d$.

By (13) we have that $m^2 = a + b + c + d \leq \sqrt{2} \cdot \sqrt{1989} < 90$. Hence $1 \leq m \leq 9$. Also, $m^4 = (a+b+c+d)^2 > a^2 + b^2 + c^2 + d^2 = 1989$, and so $m > 6$. Furthermore, since 1989 is odd, $a + b + c + d$ must be odd, and so m is odd. Thus $m = 7$ or $m = 9$.

We now eliminate the possibility that $m = 7$. If $m = 7$, then

$$\begin{aligned} n^4 + b^2 + c^2 + d^2 &= 1989 \\ n^2 + b + c + d &= 49 \end{aligned}$$

and so $2401 - 98n^2 + n^4 = (49 - n^2)^2 = (b + c + d)^2 > b^2 + c^2 + d^2 = 1989 - n^4$, and so $2n^4 - 98n^2 + 412 > 0$. This is a quadratic in the variable n^2, which has roots at (about) 45 and 5. Thus $n^2 \geq 45$ or $n^2 \leq 5$. It is easy to see that neither of these options is possible. Thus $m \neq 7$.

Thus $m = 9$, and so

$$\begin{aligned} n^4 + b^2 + c^2 + d^2 &= 1989 \\ n^2 + b + c + d &= 81 \end{aligned}$$

So $1 \leq n \leq 8$. Furthermore, $81 = n^2 + b + c + d \leq 4n^2$, and so $n > 4$. But $1989 > n^4$ and so $n \leq 6$. Thus $n = 5$ or $n = 6$.

We now eliminate the possibility that $n = 5$. If $n = 5$, then

$$\begin{aligned} b^2 + c^2 + d^2 &= 1364 \\ b + c + d &= 56 \end{aligned}$$

and of course $1 \leq d \leq c \leq b \leq 25$. Then

$$\begin{aligned} 19^2 &= (75 - (b + c + d))^2 \\ &= ((25 - b) + (25 - c) + (25 - d))^2 \\ &> (25 - b)^2 + (25 - c)^2 + (25 - d)^2 \\ &= 3 \cdot 25^2 - 50 \cdot (b + c + d) + b^2 + c^2 + d^2 \\ &= 439 \end{aligned}$$

which is impossible. Thus $n = 6$ and $a = 36$.

We now need to solve

$$\begin{aligned} b^2 + c^2 + d^2 &= 693 \\ b + c + d &= 45 \end{aligned}$$

Then clearly $b, c, d \leq 26$. Also, $693 \equiv_{16} 5$, while $x^2 \equiv_{16} \in \{0, 1, 4, 9\}$. Therefore b^2, c^2, d^2 are $0, 1, 4$ modulo 16, in no particular order. We concentrate on the odd number, which must belong to $\{1, 7, 9, 15, 17\}$. Clearly the sum of the two even numbers is congruent to 2 modulo 4, and so the odd number is congruent to 3 modulo 4. Thus it is either 7 or 15.

By some systematic experimentation we arrive at $d = 12$, $c = 15$, $b = 18$.

11.

$$\begin{aligned} C_k^n \sqrt{a_1 a_2 \cdots a_n} &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sqrt{a_1 a_2 \cdots a_n} \\ &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sqrt{a_{i_1} a_{i_2} \cdots a_{i_k}} \sqrt{\frac{a_1 a_2 \cdots a_n}{a_{i_1} a_{i_2} \cdots a_{i_k}}} \\ &\leq \sqrt{\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} a_{i_2} \cdots a_{i_k}} \sqrt{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{a_1 a_2 \cdots a_n}{a_{i_1} a_{i_2} \cdots a_{i_k}}} \\ &= \sqrt{S_k} \sqrt{S_{n-k}} \end{aligned}$$

12. We should be motivated to try and use (13) in this problem. Furthermore, the presence of the number $n-1$ (rather than the number n) suggests that we should somehow 'get rid of' one of the numbers. The manner of doing this is suggested by the 'mixed term' $2a_ia_j$. So we form a new system of $n-1$ numbers (c_i) where

$$c_1 = a_1, \ c_2 = a_2, \ \ldots, \ c_{n-2} = a_{n-2}, \ c_{n-1} = a_{n-1} + a_n$$

and then calculate:

$$\begin{aligned}(n-1)\left[A+\sum_{i=1}^{n} a_i^2\right] &< \left(\sum_{i=1}^{n} a_i\right)^2 \\ &= \left(\sum_{i=1}^{n-1} c_i\right)^2 \\ &< (n-1)\sum_{i=1}^{n-1} c_i^2 \\ &= (n-1)\left[\sum_{i=1}^{n} a_i^2 + 2a_{n-1}a_n\right]\end{aligned}$$

from which it follows that $A < 2a_{n-1}a_n$. The general result then follows because we may of course suppose that a_{n-1} and a_n are the smallest of the a_i's.

13. Notice that if we multiply the kth terms in the first and third equation we get the square of the kth term in the second equation. The same is true for the powers of A. So use of Cauchy-Schwartz is suggested. Putting $a_k = \sqrt{kx_k}$ and $b_k = \sqrt{k^5x_k}$ we get that

$$\begin{aligned}A^4 &= (A^2)^2 \\ &= \left(\sum_{k=1}^{5} k^3x_k\right)^2 \\ &= \left(\sum_{k=1}^{5} \sqrt{kx_k \cdot k^5x_k}\right)^2 \\ &\leq \sum_{k=1}^{5} kx_k \sum_{k=1}^{5} k^5x_k \\ &= A \cdot A^3 \\ &= A^4\end{aligned}$$

So the Cauchy-Schwartz inequality is actually an equality, and so there exists $\lambda \in \mathbf{R}$ such that $\lambda a_k = b_k$ i.e. $\lambda\sqrt{kx_k} = \sqrt{k^5x_k}$, or $\lambda^2 kx_k = k^5x_k$ for $1 \leq k \leq 5$.

From this it is easy to see that either all of the x_i's are zero or exactly one is non zero. These six different possibilities easily yield the solutions $A = 0, 1, 4, 9, 16, 25$.

14. Let $BC = a, AC = b, AB = c, PD = x, PE = y, PF = z$; we are required to minimise $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$. In order to use the Cauchy- Schwartz inequality, we should try to find another function of a, b, c, x, y, z that we can compare this one to. The construction tells us that $ax + by + cz$ is a constant K, namely twice the area of the triangle. Hence we get that

$$\begin{aligned}K\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right) &= (ax+by+cz)\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right) \\ &\geq \left(\sqrt{ax\frac{a}{x}}+\sqrt{by\frac{b}{y}}+\sqrt{cz\frac{c}{z}}\right)^2 \\ &= (a+b+c)^2\end{aligned}$$

which is another constant. Our quantity will be minimised when we have equality here, i.e. when

$$\begin{aligned}ax &= \lambda\frac{a}{x} \\ by &= \lambda\frac{b}{y} \\ cz &= \lambda\frac{c}{z}\end{aligned}$$

for some $\lambda \in \mathbf{R}$. It follows trivially that $x^2 = y^2 = z^2 = \lambda$, so $x = y = z$, and P is the incentre of the triangle.

15. The proof will be by induction on the number of members in the set. If $|S| = 1$ then $|S_x| = |S_y| = |S_z| = 1$ and so the result is trivially true. Suppose now the result holds true for any set S such that $|S| < n$. We now show that it holds true if $|S| = n$.

Take such a set and consider the collection of planes which are parallel to the xy-plane. If possible, choose such a plane that divides the set S up into two parts, the part S^1 below the plane and the part S^2 above the plane. It is required that none of the points of S actually lie in the plane.

The only way that such a construction would be impossible is if all the points of S lay in one particular horizontal plane. If so, then we simply 'rotate' our perspective and again look for a suitable plane. If this is again impossible, then all the points lie on a single line, and we 'rotate' again. If the construction is again impossible then S must be a single point : and we have already dealt with that case.

So suppose that the set S has been divided up into S^1 and S^2 as described. Then $|S_z| \leq |S_z^1| + |S_z^2|$, $|S_x| = |S_x^1| + |S_x^2|$ and $|S_y| = |S_y^1| + |S_y^2|$. (Do you understand why there is one inequality and two equalities here?) Of course, $|S_x^i| \leq |S_x|$,

$|S_y^i| \le |S_y|$ and $|S_z^i| \le |S_z|$ for $i = 1, 2$. Hence

$$\begin{aligned}
|S|^2 &= \left(|S^1| + |S^2|\right)^2 \\
&\le \left(\sqrt{|S_x^1| \cdot |S_y^1| \cdot |S_z^1|} + \sqrt{|S_x^2| \cdot |S_y^2| \cdot |S_z^2|}\right)^2 \\
&\le \left(\sqrt{|S_x^1| \cdot |S_y^1|} + \sqrt{|S_x^2| \cdot |S_y^2|}\right)^2 \cdot |S_z| \\
&\le \left(|S_x^1| + |S_x^2|\right) \cdot \left(|S_y^1| + |S_y^2|\right) \cdot |S_z| \\
&= |S_x| \cdot |S_y| \cdot |S_z|
\end{aligned}$$

6.5 Jensen's Inequality

1.

$$\begin{aligned}
\frac{x^2+y^2+z^2}{3} &\ge \left(\frac{x+y+z}{3}\right)^2 = \frac{1}{9} \\
\Rightarrow x^2+y^2+z^2 &\ge \frac{1}{3} \\
\Rightarrow 1 &= 1 \cdot 1 \\
&= (x+y+z) \cdot (x+y+z) \\
&= x^2+y^2+z^2+2xy+2yz+2zx \\
&\ge \frac{1}{3} + 2(xy+yz+zx) \\
\Rightarrow \frac{2}{3} &\ge 2(xy+yz+zx) \\
\Rightarrow \frac{1}{3} &\ge xy+yz+zx
\end{aligned}$$

2. The sine graph is concave, the cosecant graph convex. The cosine graph is concave until 90° and then becomes convex. The graph is continuous throughout. (A point where the graph is continuous and changes from concavity to convexity or *vise versa* is called a point of inflection.) Similarly for the cotangent function, although here concavity and convexity are reversed.

   Both the tangent and secant functions are convex before 90°, there are discontinuities at 90°, and then are concave.

3. The sine function is concave on $(0°, 180°)$ and the cosecant function is convex on the interval $(0°, 180°)$, so it follows that

$$\frac{\sin a + \sin b + \sin c}{3} \le \sin(\frac{a+b+c}{3}) = \sin 60° = \frac{\sqrt{3}}{2}$$

   and similarly

$$\frac{\csc a + \csc b + \csc c}{3} \ge \csc(\frac{a+b+c}{3}) = \csc 60° = \frac{2}{\sqrt{3}}$$

The cosine and cotangent functions present special problems because they change concavity at 90°. Nevertheless we are still able to make clever use of Jensen's inequality.

Firstly, the cosine function is concave on $(0°, 90°)$. From this it is easy to see immediately that if the triangle is acute angled then $\cos a + \cos b + \cos c \le \frac{3}{2}$. Now suppose that $c \ge 90°$. Then :-

$$\begin{aligned}
\cos a + \cos b + \cos c &= 2 \cdot \frac{\cos a + \cos b}{2} + \cos c \\
&= 2 \cdot \frac{\cos a + \cos b}{2} - \cos(a+b) \\
&\le 2 \cdot \cos(\frac{a+b}{2}) - \cos(a+b) \\
&= 2 \cdot \cos(\frac{a+b}{2}) + 1 - 2 \cdot \cos^2(\frac{a+b}{2}) \\
&= 2C + 1 - 2C^2 \\
&= -2(C^2 - C + \frac{1}{4}) + \frac{3}{2} \\
&= -2(C - \frac{1}{2})^2 + \frac{3}{2} \\
&\le \frac{3}{2}
\end{aligned}$$

where of course $C = \cos(\frac{a+b}{2})$.

Similarly, since the cotangent function is convex on $(0°, 90°)$, if the triangle is acute angled then $\cot a + \cot b + \cot c \ge \sqrt{3}$. Now suppose that $c \ge 90°$. Then :-

$$\begin{aligned}
\cot a + \cot b + \cot c &= 2 \cdot \frac{\cot a + \cot b}{2} + \cot c \\
&= 2 \cdot \frac{\cot a + \cot b}{2} - \cot(a+b) \\
&\ge 2 \cdot \cot(\frac{a+b}{2}) - \cot(a+b) \\
&= 2 \cdot \cot(\frac{a+b}{2}) - \frac{\cot^2(\frac{a+b}{2}) - 1}{2\cot(\frac{a+b}{2})} \\
&= 2C - \frac{C^2 - 1}{2C} \\
&= \frac{3}{2C}\left[C^2 + \frac{1}{3}\right] \\
&= \frac{3}{2C}\left[(C - \frac{1}{\sqrt{3}})^2 + 2C\frac{1}{\sqrt{3}}\right] \\
&\ge \sqrt{3}
\end{aligned}$$

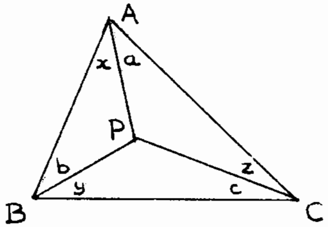
where of course $C = \cot(\frac{a+b}{2})$.

It should by now be easy for the reader to verify that $\tan a + \tan b + \tan c \geq 3\sqrt{3}$ and $\sec a + \sec b + \sec c \geq 6$ for acute angled triangles. By considering isosceles triangles with an obtuse angle just greater than 90°, one sees that these functions are not bounded from below.

4. Let the angles of the triangle be 2α, 2β, 2γ. Then since the incircle has radius 1, the semiperimeter is $\cot\alpha + \cot\beta + \cot\gamma$. Since $0° < \alpha, \beta, \gamma \leq 90°$ we have from a previous exercise that this quantity is greater than or equal to $3\sqrt{3}$ and is minimised when $\alpha = \beta = \gamma = 30°$ i.e. when the triangle is equilateral.

5. (The solution presented below is due to Ravi Vakil. It is far quicker, but perhaps less obvious, than the solution given by the proposer.)

   Let P be any point, and denote $< CAP$, ABP and BCP by a, b and c respectively. Denote $< PAB$, $< PBC$ and $< PCA$ by x, y and z respectively.



Assume for a contradiction that all of a, b and c are greater than 30°. Then $x + y + z < 90°$ and so each of a, b and c is less than 120°. It follows from this (inspect a sine graph if necessary) that $\sin a, \sin b, \sin c > 0.5$. Now

$$\frac{\sin a}{\sin z}\cdot\frac{\sin b}{\sin x}\cdot\frac{\sin c}{\sin y} = \frac{CP}{AP}\cdot\frac{AP}{PB}\cdot\frac{PB}{CP} = 1$$

that is, $\sin a \cdot \sin b \cdot \sin c = \sin x \cdot \sin y \cdot \sin z$. Therefore

$$\begin{aligned} 0.5 &= \sqrt[3]{0.5\cdot 0.5\cdot 0.5} \\ &< \sqrt[3]{\sin a\cdot\sin b\cdot\sin c} \\ &< \sqrt[3]{\sin x\cdot\sin y\cdot\sin z} \\ &\leq \frac{\sin x + \sin y + \sin z}{3} \\ &\leq \sin\left(\frac{x+y+z}{3}\right) \\ &< \sin\left(\frac{90°}{3}\right) \\ &= 0.5 \end{aligned}$$

Here we have used the Arithmetic-Geometric mean inequality and then Jensen's inequality for the sine function.

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