WMC 2019 Undergraduate Final Round Solutions

Section A

1. **25**

   This can be done by the Sieve of Eratosthenes.

2. **175**

   A perfect square is a multiple of 54 if and only if it is in the form $324N^2$ for some integer N. Then we must find the largest N such that $324N^2 < 10^7$ which is $N = 175$.

3. **5416**

   We partition these numbers by the first time a 5 or a 6 appears. There are 2000 numbers that begin with a 5 or a 6. There are 2000 numbers that begin with a 5 or a 6. There are $7 \times 2 = 1400$ numbers that have a 5 or 6 appear first in the hundreds digit. There are $7 \times 8 \times 2 \times 10 = 1120$ that have a 5 or a 6 first occur as the tens digit. Finally there are $7 \times 8 \times 8 \times 2 = 896$ have a 5 or a 6 first appear as the units digit. So in total there are $2000 + 1400 + 1120 + 896 = 5416$ such numbers.

4. $\boldsymbol{\frac{25\pi}{2}}$

   The integral defines a semi-circle of radius 5.

5. $\boldsymbol{\frac{2}{2019}}$

   Writing out the first few terms suggests that $f(n) = \frac{f(1)}{\binom{n+1}{2}}$ which we show by induction.
   We have that the base case holds. Now we induct

$$\begin{aligned} f(1) + f(2) + \cdots + f(n) + f(n+1) &= (n+1)^2 f(n+1) \\ \implies n^2 f(n) + f(n+1) &= (n+1)^2 f(n+1) \\ \implies n^2 f(n) &= n(n+2) f(n) \\ \implies f(n+1) &= \frac{f(1)}{\binom{n+2}{2}} \end{aligned}$$

   Then letting $n = 2019$ and $f(1) = 2019$ gives the result.

6.

7. $\boldsymbol{P_4}$

   Movement towards P_4 is always moving at least 4 goal posts and movement away from P_4 is always moving towards at most 3.

8. **10**

The angles of a regular polygon with n sides are equal to $\frac{180^\circ(n-2)}{n}$. In particular the angles of a regular pentagon are equal to 108° and so the angles of the polygon formed by sticking the pentagons together is $360^\circ - 2 \times 108^\circ = 144^\circ$. Then now if N is the number of rings we must have $\frac{180^\circ(N-2)}{N} = 144 \implies N = 10$.

9. **1458**

Observe that the sum of the children's ages must be a multiple of 9 since the number formed by putting the ages together is a multiple of 81 and so is a multiple of 9. So the possible age sums are 9, 18, 27 and 36. We cannot go any higher since each age is a single digit so it is at most 9. It remains then to deal with each of these cases.

10. **142857**

The trick here is to consider the decimal expansion of $\frac{1}{7}$. Since 7 is prime its reciprocal must have period 6. In particular $\frac{1}{7} = 0.142857142857\ldots$. So our number turns out to be these six digits 142857.

Section B

11. The probability that passenger 2019 sits in seat 2019 is $\frac{1}{2}$.
Passenger 2019 will seat in their assigned seat if and only if someone sits in seat 1 before someone sits in seat 2019. As every passenger except the last can only sit in these sits if they are making choice, it is sufficient to observe that all such choices are equally likely to be in seat 1 or seat 2019.

12. $\frac{2}{3}$.
Consider the top three players, and consider the first two times two of these players meet up. By symmetry this meeting is equally likely to contain any two of these three top players. If the top player meets the second best player, she will go on to defeat the third best player later in the tournament. If the top player meets the third best player first, she will eliminate the third best player immediately. Finally if the second best player meets the second best player, the second best player will eliminate the third best player and prevent the top player from eliminating the third best player.