GEOMETRY

for the

OLYMPIAD ENTHUSIAST

Bruce Merry

Introduction

The South African Mathematical Society (SAMS) has the responsibility for selecting teams to represent South Africa at the Pan African Mathematics Olympiad (PAMO) and the International Mathematical Olympiad (IMO).

Team selection begins with the Old Mutual Mathematical Talent Search, a self-paced correspondence course in problem-solving that starts afresh in January each year. The best performers in the Talent Search are invited to attend mathematical camps in which they learn specialised problem-solving skills and write challenging Olympiad-level papers. Since the Pan African Maths Olympiad is not quite as daunting as the International version, the tradition has evolved that South African PAMO teams consist of students who have not previously been selected for an IMO team. The Inter-Provincial Mathematical Olympiad and the South African Mathematics Olympiad are closely linked with the PAMO and IMO selection programme.

To provide background reading for the Talent Search, the South African Mathematical Society has published a series of Mathematical Olympiad Training Notes that focus on mathematical topics and problem-solving skills needed in mathematical competitions and Olympiads. Six booklets have appeared to date:

- *The Pigeon-hole Principle*, by Valentine Goranko
- *Topics in Number Theory*, by Valentin Goranko
- *Inequalities for the Olympiad Enthusiast*, by Graeme West
- *Graph Theory for the Olympiad Enthusiast*, by Graeme West
- *Functional Equations for the Olympiad Enthusiast*, by Graeme West
- *Mathematical Induction for the Olympiad Enthusiast*, by David Jacobs

Bruce Merry's *Geometry for the Olympiad Enthusiast* is an important and welcome addition to this series.

Though their primary target is the development of high-level problem-solving skills, these booklets can be read by anybody interested in the mathematics just beyond the high school curriculum. They are therefore particularly useful to teachers looking for enrichment material, and students who plan to study mathematics at university level and would like more of a challenge than the school curriculum provides.

For more information, write to

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John Webb
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Geometry for the Olympiad Enthusiast

Bruce Merry

A booklet in this series was last published in 1996, and the series has been somewhat dormant since. Geometry has long been a gap in this series, and eventually I decided to address this gap. I started writing this booklet in December 2000. It was then put aside for three years, while I focused on my studies. In December 2003 I finally returned to finish the rather delayed project.

This booklet is primarily about classical, or Euclidean, geometry. Trigonometry is used as a tool, but is not explored in great depth, and coordinate geometry barely puts in an appearance. While tackling the exercises and geometry problems in general, one should remember that trigonometry and coordinate geometry are powerful tools. I simply did not have much to say about them.

The booklet assumes a knowledge of high-school geometry. If you have not completed the high-school syllabus, it would be a good idea to first find a textbook and work through both the theory and the exercises. The proofs included here are somewhat terse and you may need to fill in a few details yourself.

Some important results are left as problems, so you should at the very least read the problems (although you really should attempt to solve them, as well). The positioning of problems in the book is a good indicator of how you are expected to tackle them, although of course there are usually other solutions. There are two types of problems: exercises that deal very specifically with the topic in hand, and real olympiad problems. The olympiad problems are labelled with a star ($\star$). The exercises are generally easier than the olympiad problems, but some of them are quite challenging.

I would like to thank Dirk Laurie for writing his Geomplex diagram drawing package. This book would not have been possible without it. I would also like to thank Mark Berman, whose flair for geometry has always inspired me to find elegant solutions.

Contents

1 Techniques

Geometry is unlike many of the other areas of olympiad mathematics, requiring more intuition and less algebra. Nevertheless, it is important to do the basic groundwork as otherwise your intuition has nothing with which to work.

Here are some suggestions on ways to approach a geometry problem.

- Draw a quick diagram so that you can visualise the problem.
- Draw a neat and accurate diagram — this will often reveal additional facts which you could then try to prove.
- Draw a deliberately incorrect diagram (this could be your initial diagram), so that you don't accidentally assume the result because you referred to your accurate diagram (this is particularly important if you are proving concurrency or collinearity).
- It is very important to do as much investigation as you can. Try to relate as many angles and line segments as you can, even if you have several variables. Then look for similar or congruent triangles, parallel lines and so on. This on its own can be enough to solve some easier problems without even having to think.
- There are many approaches to attack geometry problems e.g. Euclidean geometry, coordinate geometry, complex numbers, vectors and trigonometry. Think about applying all the ones that you know to the problem and deciding which ones are most likely to work. Be guided by what you are asked to prove: for example, if you are asked to prove that two lines are parallel then coordinate geometry might work well, but if the problem involves lots of related angles then trigonometry may be a better approach.
- Don't be afraid to get your hands dirty with trigonometry, coordinate geometry or algebra. While such solutions might not be as "cool" as solutions that require an inspired construction, they are often easier to find and score the same number of points. However, doing as much as possible with Euclidean geometry first can make the equations simpler.
- Look for constructions that will give you similar triangles, special angles or allow you to restate the problem in a simpler way. For example, if you are asked to prove something about the sum of two lengths, try making a construction that places the two lengths end to end so that you only have to prove something about the length of a single line.

- Assume that the result is true, and see what follows from this. This may lead you to intermediate results which you can then try to prove.

- Always check that you haven't omitted any cases such as obtuse angles or constructions that are impossible in certain cases (for example, you can't take the intersection point of two lines if they are parallel). This booklet does a terrible job of this, because the special cases are almost always trivial. I'm lazy, the duplication costs of this booklet are high, the rainforests are dying, and this is not a competition. In a competition, you can expect to lose marks if your proof does not work in all cases.

2 Terminology and notation

There is some basic terminology for things that share some property. Concurrent lines pass through a common point, and collinear points lie on a common line. Concyclic points lie on a common circle; note that "A, B, C and D are concyclic" does not have the same meaning as "$ABCD$ is a cyclic quadrilateral", since the latter implies that the points lie in a particular order around the circle. Concentric circles have a common centre.

The humble triangle has possibly the richest terminology and notation. There are numerous "centres", generally the point of concurrency of certain lines, and a few have corresponding circles.

incentre The centre of the incircle (inscribed circle); the point of concurrency of the internal angle bisectors

circumcentre The centre of the circumcircle (circumscribed circle); the point of concurrency of the perpendicular bisectors

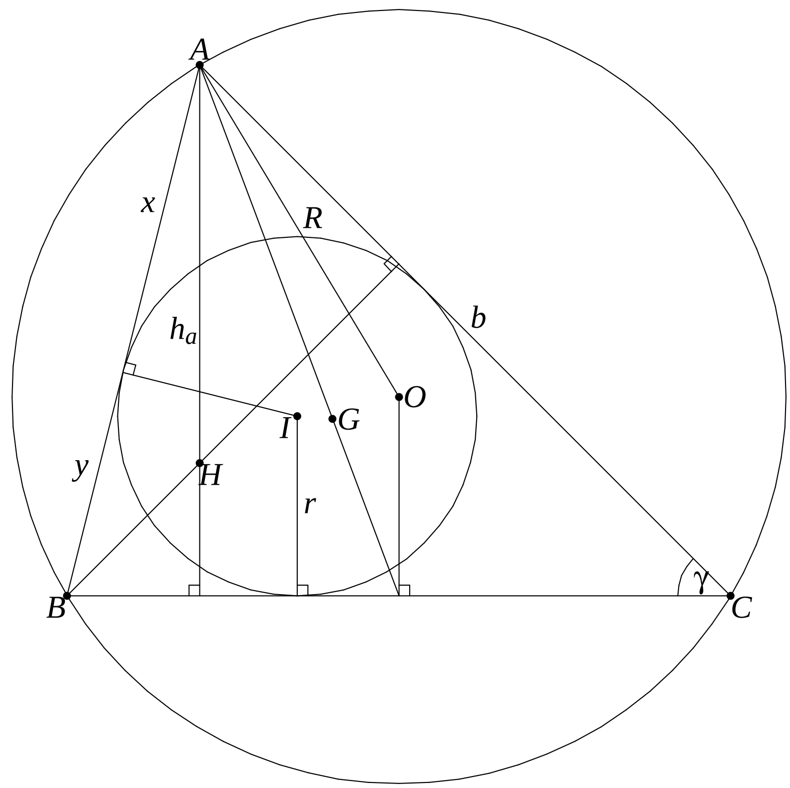
excircle The centre of an excircle (escribed circle); the point of concurrency of two external and one internal angle bisector

orthocentre The point of concurrency of the altitudes

centroid The point of concurrency of the medians (lines from a vertex to the midpoint of the opposite side)

Most of these terms should be familiar from high-school geometry. An unfamiliar term is a *cevian*: this is any line joining a vertex to the opposite side.

For this booklet (particularly section 6), we also introduce a lot of notation for triangles. Some of this is standard or mostly standard while some is not; you are advised to define any of these quantities in proofs, particularly K, x, y and z.



I the incentre

I_A the excentre opposite A

O the circumcentre

G the centroid

H the orthocentre

a the side opposite vertex A (similarly for B and C)

s the semiperimeter, $\frac{a+b+c}{2}$

x the tangent from A to the incircle, $\frac{-a+b+c}{2} = s - a$ (similarly for y and z)

R the radius of the circumcircle (circumradius)

r the radius of the incircle (inradius)

r_a the radius of the excircle opposite A

h_a the height of the altitude from A to BC

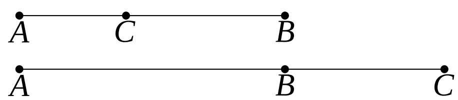
α the angle at A (similarly for β and γ)

K the area of the triangle

We also use the notation $|\triangle ABC|$ (or just $|ABC|$) to indicate the area of $\triangle ABC$.

3 Directed angles, line segments and area

In classical geometry, most quantities are undirected. That means that if you measure them in the opposite direction, they have the same value ($AB = BA$, $\angle ABC = \angle CBA$, and $|\triangle ABC| = |\triangle CBA|$). Most of the time this is a reasonable way of doing things. However, it occasionally has disadvantages. For example, if you know that A, B and C are collinear, and $AB = 5$, $BC = 3$, then what is AC? It could be either 2 or 8, depending on which way round they are on the line. The same problem arises when adding angles or areas.



Normally these situations are not important, because it is clear from a diagram which is correct. However, sometimes there are many different ways to draw the diagram, leading to a proof with many different cases. Another way to solve the problem is to treat the quantities as having a sign, indicating the direction. So now if you are told that $AB = 5$, $BC = 3$ then you can be sure that $AC = AB + BC = 8$. This is because both have the same sign, and hence are in the same direction. If C lay between A and B, then $AB = 5$, $BC = -3$ and so $AC = AB + BC = 2$. It could also be that $AB = -5$, $BC = 3$; the positive direction is generally arbitrary but must be consistent. What is important is that no matter in what order A, B and C lie, the equation $AC = AB + BC$ holds.

Directed line segments have somewhat limited use, because it only makes sense to compare lines that are parallel. Generally they are used when dealing with ratios or products of collinear line segments (see Menelaus' Theorem (6.3), for example). Directed angles and directed area are more often used.

A directed angle $\angle ABC$ is really a measure of the angle between the two lines AB and BC. Conventionally, it is the amount by which AB must be rotated anti-clockwise to line up with BC. One effect of this is that while normal angles have a range of 360°, directed angles only have a range of 180°! This is because rotating a line by 180° leaves it back where it started, so 180° is equivalent to 0°. To indicate this, equivalent angles are sometimes written $\angle ABC \equiv \angle DEF$ rather than $\angle ABC = \angle DEF$. This limitation occasionally has disadvantages, and in particular it is not generally possible to combine trigonometry with directed angles (since the sin and cos functions only repeat every 360°). This is made up for by the special properties that directed angles do have:

1. $\angle AMC \equiv \angle AMB + \angle BMC$;
2. $\angle AXY \equiv \angle AXZ$ iff X, Y, Z are collinear
3. $\angle XYZ \equiv 0^\circ$ iff X, Y, Z are collinear
4. $\angle ABC + \angle BCA + \angle CAB \equiv 0$;
5. $\angle PQS \equiv \angle PRS$ iff P, Q, R and S are concyclic.

Property 1 is simply the basis of directedness: the relative positions don't matter. Property 2 is trivial if Y and Z lie on the same side of X, and the fact that adjacent angles add up to 180° if not. Property 3 just restates the fact that rotating a line onto itself leads to no rotation. Property 4 is the result that angles in a triangle add up to 180°, but also brings in the fact that the three angles are either all clockwise or all anti-clockwise. Property 5 is the really interesting one: it is *simultaneously* the same segment theorem and the alternate segment theorem, depending on the ordering of the points on the circle. The problem below illustrates why having a single theorem can be so important.

Directed areas are used even less often than directed angles and line segments, but are sometimes useful when adding areas to compute the area of a more complex shape. Conventionally, a triangle ABC has positive area if A, B and C are arranged in anti-clockwise order, and negative if they are arranged in clockwise order.

Exercise 3.1. *Three circles, Γ_1, Γ_2 and Γ_3 intersect at a common point O. Γ_1 and Γ_2 intersect again at X, Γ_2 and Γ_3 intersect again at Y, and Γ_3 and Γ_1 intersect against at Z. A is a point on Γ_1 which does not lie on Γ_2 or Γ_3. AX intersects Γ_2 again at B, and BY intersects Γ_3 again at C. Prove that A, Z and C are collinear.*

Exercise 3.2 (Simpson Line). *Perpendiculars are dropped from a point P to the sides of $\triangle ABC$ to meet BC, CA, AB at D, E, F respectively. Show that D, E and F are collinear if and only if P lies on the circumcircle of $\triangle ABC$.*

You will find that directed angles in particular play a large role in the theorems in this book, and they are introduced early on for this purpose. Do not be led to believe that directed angles are so wonderful that they should be used for all problems: theorems try to make very general statements and use directed angles for generality, but most problems are constrained so that normal angles are adequate (e.g. points inside triangles or acute angles). Normal angles are easier to work with simply because one does not need to think about whether to write $\angle ABC$ or $\angle CBA$.

4 Trigonometry

Trigonometry is seldom required to solve a problem. After all, trigonometry is really just a way of reasoning about similar triangles. However, it is a very powerful reasoning tool, and if applied correctly can replace a page full of unlikely and ungainly constructions with a few lines of algebra. If applied incorrectly, however, it can have the opposite effect.

The first thing to do before applying any trigonometry is to reduce the number of variables to the minimum. Then choose the variables that you want to keep very carefully. The compound angle formulae below make it easy to expand out many trig expressions, but if you have chosen the wrong variables to start with the task is almost impossible.

The following angle formulae are invaluable in manipulating trigonometric expressions. In the formulae below, a $\mp$ indicates a sign that is opposite to the sign chosen in a $\pm$.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{4.1}$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{4.2}$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \tag{4.3}$$
$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot A \pm \cot B} \tag{4.4}$$
$$\sin A \sin B = [\cos(A-B) - \cos(A+B)]/2 \tag{4.5}$$
$$\sin A \cos B = [\sin(A-B) + \sin(A+B)]/2 \tag{4.6}$$
$$\cos A \cos B = [\cos(A-B) + \cos(A+B)]/2 \tag{4.7}$$
$$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right) \tag{4.8}$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{4.9}$$

$$\cos A - \cos B = 2\sin\left(\frac{B+A}{2}\right)\sin\left(\frac{B-A}{2}\right) \tag{4.10}$$

You don't need to memorise any of these other than the first three, because all the others can be obtained from these with simple substitutions. You should be aware that these transformations exist and know how to derive them, so that you can do so in an olympiad if necessary (see the exercises).

You can also use these to derive other formulae; for example, you can calculate $\sin n\theta$ and $\cos n\theta$ in terms of $\sin\theta$ and $\cos\theta$ fairly easily (for small, known values of n).

Exercise 4.1. *Prove equations* (4.4) *to* (4.10).

Exercise 4.2. *In a $\triangle ABC$ (which is not right-angled), prove that*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

4.1 The extended sine rule

The standard Sine Rule says that

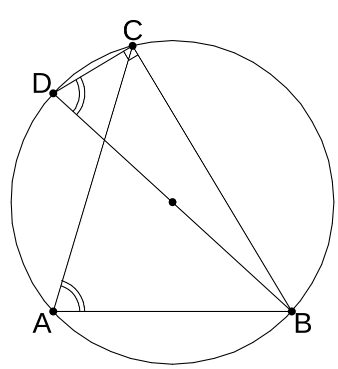
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Theorem 4.1 (Extended Sine Rule). *In a triangle ABC,*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the radius of the circumcircle.

Proof. Construct point D diametrically opposite B in the circumcircle of $\triangle ABC$. Then $\alpha = \angle CDB$ or $180^\circ - \angle CDB$ and $\angle BCD = 90^\circ$. It follows that $\frac{a}{\sin\alpha} = \frac{BC}{BC/BD} = 2R$, and similarly for $\frac{b}{\sin\beta}$ and $\frac{c}{\sin\gamma}$.



□

Exercise 4.3. *In a circle with centre O, AB and CD are diameters. From a point P on the circumference, perpendiculars PQ and PR are dropped onto AB and CD respectively. Prove that the length of QR is independent of the position of P.*

5 Circles

5.1 Cyclic quadrilaterals

A cyclic quadrilateral is a quadrilateral that can be inscribed in a circle. There are several results related to the angles of a cyclic quadrilateral that are covered in high school mathematics and which will not be repeated here. These results are still very important, and cyclic quadrilaterals appear in many unexpected places in olympiad problems.

Exercise 5.1 ($\star$). *Let $\triangle ABC$ have orthocentre H and let P be a point on its circumcircle. Let E be the foot of the altitude BH, let $PAQB$ and $PARC$ be parallelograms, and let AQ meet HR in X.*

(a) Show that H is the orthocentre of $\triangle AQR$.

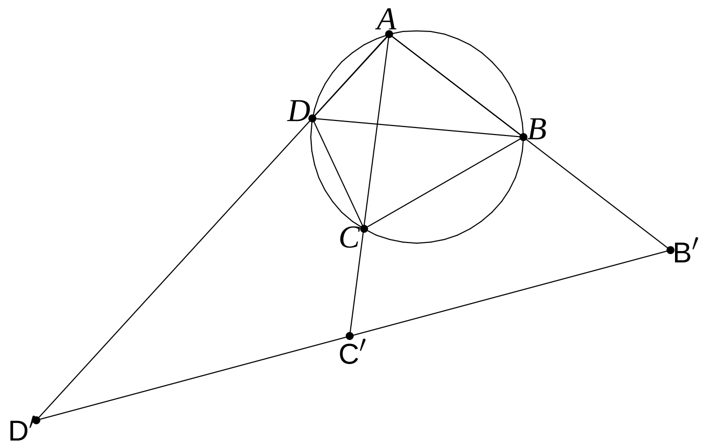
(b) Hence, or otherwise, show that EX is parallel to AP.

A result that is not normally taught in school is Ptolemy's Theorem. It is mainly useful if you have only one or two cyclic quadrilaterals, and lengths play a major role in the problem. It is also very useful when some more is known about the lengths. Equal lengths are particularly helpful as they can divide out of the equation.

Theorem 5.1 (Ptolemy's Theorem). *If ABCD is a cyclic quadrilateral, then*

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

Proof.



Choose an arbitrary constant K and construct B', C' and D' on AB, AC and AD respectively such that $AB \cdot AB' = AC \cdot AC' = AD \cdot AD' = K$.

Now consider $\triangle ABC$ and $\triangle AC'B'$. The angle at A is common and $\frac{AB}{AC'} = \frac{K/AB'}{K/AC} = \frac{AC}{AB'}$ and therefore the triangles are similar. It follows similarly that $\triangle ABD \,|||\, \triangle AD'B'$ and $\triangle ACD \,|||\, \triangle AD'C'$. Hence $\angle B'C'D' = \angle ABC + \angle ADC = 180^\circ$ i.e. $B'C'D'$ is a straight line. From the similar triangles, we have $BC = B'C' \cdot \frac{AB}{AC'} = \frac{B'C' \cdot AB \cdot AC}{K}$, and similarly for CD and BD. Therefore

$$\begin{aligned} AC \cdot BD &= \tfrac{B'D'}{K}(AB \cdot AC \cdot AD) \\ &= (\tfrac{B'C'}{K} + \tfrac{C'D'}{K})(AB \cdot AC \cdot AD) \\ &= AB \cdot CD + AD \cdot BC \end{aligned}$$

This result relies on the fact that $B'C'D'$ is a straight line. If we had used a non-cyclic quadrilateral, this would not have been the case. This shows that the converse of Ptolemy's Theorem is also true. In fact the triangle inequality in $\triangle B'C'D'$ leads to *Ptolemy's Inequality*, which says that $AC \cdot BD \leq AB \cdot CD + AD \cdot BC$ for any quadrilateral $ABCD$, with equality precisely for cyclic quadrilaterals. □

Exercise 5.2. *Triangle ABC is equilateral. For any point P, show that $AP + BP \geq CP$ and determine when equality occurs.*

5.2 The Simpson line

The Simpson line was covered as exercise 3.2, but to emphasise its importance the statement is repeated here. A handy corollary is that the feet of perpendiculars from a point on the circumcircle cannot all meet the sides internally — which can limit the number of cases you need to consider.

Theorem 5.2 (The Simpson line). *Perpendiculars are dropped from a point P to the sides of $\triangle ABC$ to meet BC, CA, AB at D, E, F respectively. Show that D, E and F are collinear if and only if P lies on the circumcircle of $\triangle ABC$.*

This was exercise 3.2, so no proof is provided here.

Exercise 5.3. *From a point E on a median AD of $\triangle ABC$ the perpendicular EF is dropped to BC, and a point P is chosen on EF. Then perpendiculars PM and PN are drawn to the sides AB and AC.*

Now, it is most unlikely that M, E and N will lie in a straight line, but in the event that they do, prove that AP bisects $\angle A$.

5.3 Power of a point

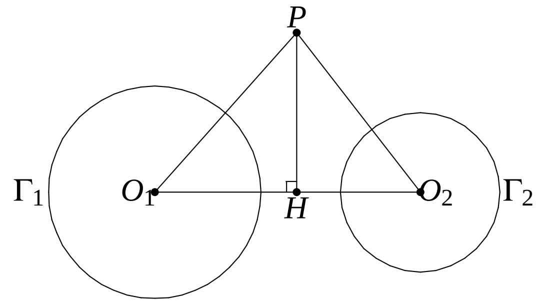
This section is based on the fact that if chords AB and CD of a circle intersect at a point P, then $PA \cdot PB = PC \cdot PD$ (even if P lies outside the circle). This is easily shown using similar triangles.

Consider fixing a point P and circle Γ and considering all possible chords AB that pass through P. Since $PA \cdot PB$ is equal for every pair of chords AB, it is equal for *all* such chords. This value is said to be the power of P with respect to Γ. The line segments are considered to be directed (see section 3), so P is negative inside the circle and positive outside of it. In fact by considering the chord that passes through O, the centre of Γ, it can be seen that the power of P is $d^2 - r^2$, where $d = OP$ and r is the radius of Γ. If P lies outside the circle then this also equals the square of the length of the tangent from P to Γ.

It is sometimes useful to know that the converse of the above result is true i.e. if $PA \cdot PB = PC \cdot PD$, where AB and CD pass through P, then A, B, C and D are concyclic (but only if using directed line segments).

5.3.1 The radical axis

Consider having two circles instead of one. What is the set of points which have the same power with respect to both circles? If the circles are concentric then no point will have the same power (since d will be the same and r different for every point), but the situation is less clear in general.



Consider two circles Γ_1 and Γ_2 with centres O_1 and O_2 with radii r_1 and r_2 respectively. Let P be a point which has equal powers with respect to Γ_1 and Γ_2, and let H be the foot of the perpendicular from P onto O_1O_2. Then

$$
\begin{aligned}
& & O_1P^2 - r_1^2 &= O_2P^2 - r_2^2 & (5.1)\\
&\Longleftrightarrow & O_1H^2 + HP^2 - r_1^2 &= O_2H^2 + HP^2 - r_2^2 & (5.2)\\
&\Longleftrightarrow & O_1H^2 - r_1^2 &= O_2H^2 - r_2^2 & (5.3)\\
&\Longleftrightarrow & O_1H^2 - r_1^2 &= (O_2O_1 - HO_1)^2 - r_2^2 & (5.4)\\
&\Longleftrightarrow & 2 \cdot HO_1 \cdot O_2O_1 &= {O_2O_1}^2 + r_1^2 - r_2^2 & (5.5)
\end{aligned}
$$

We have eliminated P from the equation! In fact (5.3) shows that P has equal powers with respect to the circles iff H does. If $O_1O_2 \neq 0$ then we have a linear equation in HO_1 and so there is exactly one possibility for H (we are using directed line segments, so HO_1 uniquely determines H). Thus the locus of P is the line through H perpendicular to O_1O_2. This line is known as the *radical axis* of Γ_1 and Γ_2.

If the two circles intersect, the radical axis is easy to construct. The points of intersection both have zero power with respect to both circles, so both points lie on the radical axis. So the radical axis is simply the line through them.

Exercise 5.4. *Two circles are given. They do not intersect and neither lies inside the other. Show that the midpoints of the four common tangents are collinear.*

5.3.2 Radical centre

What happens when we consider three circles (say Γ_1, Γ_2 and Γ_3) instead of two? Firstly consider the case where the centres are not collinear. Then the radical axis of Γ_1 and Γ_2 will meet the radical axis of Γ_2 and Γ_3 at some point, say X (they will not be parallel because a radical axis is perpendicular to the line between the centres of the circles). Then from the definition of a radical axis, X has the same power with respect to all three circles and so it also lies on the radical axis of Γ_1 and Γ_2. The fact that the three radical axes are concurrent at a point (known as the *radical centre*) can be used to solve concurrency problems.

If, however, the three centres are collinear, then all three radical axes are parallel. If they all coincide then all points on the common axis have equals powers with respect to the three circles; if not then no points do.

Exercise 5.5. *Show how to construct, using ruler and compass, the radical axis of two non-intersecting circles.*

Exercise 5.6 ($\star$). *Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.*

6 Triangles

6.1 Introduction

A triangle would seem to be almost the simplest possible object in geometry, second only to the circle. It has only has two true degrees of freedom, since scaling a triangle up or down does not affect its properties. Yet the humble triangle contains an enormous amount of mathematics — in fact too much to fully explore here.

6.2 Tangents to the incircle

Let the lengths of the tangents to the incircle from A, B and C be x, y and z. Since $a = y+z$, $b = z+x$ and $c = x+y$, we can solve for x, y and z and get

$$x = \frac{-a+b+c}{2}, \quad y = \frac{a-b+c}{2}, \quad z = \frac{a+b-c}{2}.$$

This is the same notation that is introducted in section 2.

Exercise 6.1. *Determine the lengths of the tangents from B and C to the excircle opposite A.*

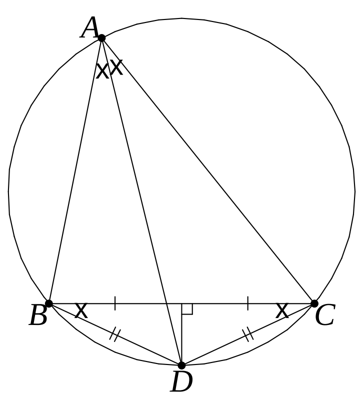
6.3 Triangles within triangles

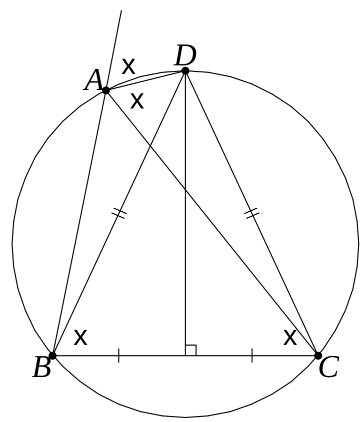
There are specific names given to certain triangles formed from points of the original triangle:

- The *medial* triangle has the midpoints of the original sides as its vertices.
- The *orthic* triangle has the feet of the altitudes as its vertices.
- A *pedal* triangle is the triangle formed by the feet of perpendiculars dropped from some point onto the three sides. If the point is the orthocentre, then this is the orthic triangle (and in fact some people use the term "pedal triangle" to refer to the orthic triangle).

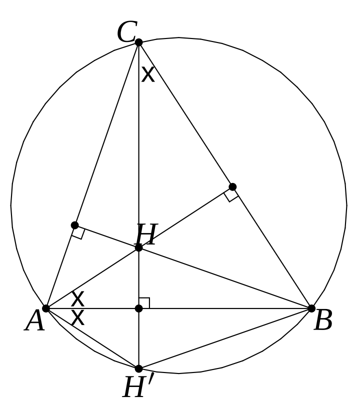
6.4 Points on the circumcircle

Apart from the vertices, there are a few other points that are known to lie on the circumcircle. The first is the intersection point of a perpendicular bisector and the corresponding angle bisector. This is easily shown by taking the intersection of the perpendicular bisector and the circumcircle, which divides an arc (say BC) into two equal parts which subtend equal angles at A. This is also true (although less well known) in the case where the *external* angle bisector is used.





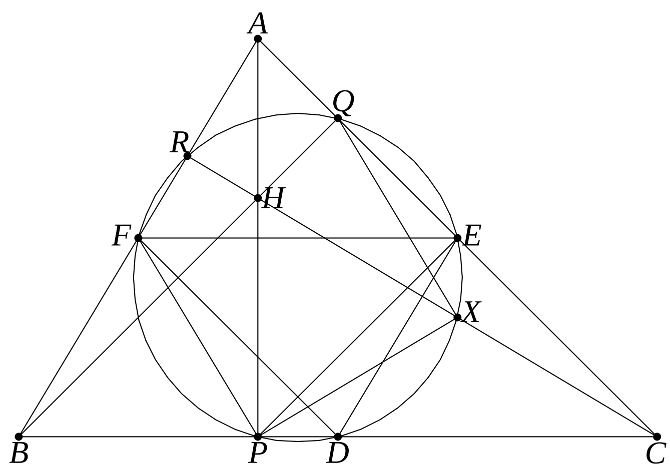
The second group of points that are known to lie on the circumcircle are the reflections of H (the orthocentre) in each of the three sides. This is an exercise in angle chasing, using the known results about the angles in cyclic quadrilaterals.



Exercise 6.2. *A rectangle $HOMF$ has $HO = 23$ and $OM = 7$. Triangle ABC has orthocentre H and circumcentre O. The midpoint of BC is M and F is the foot of the altitude from A. Determine the length of side BC.*

6.5 The nine-point circle

A rather interesting circle that arises in a triangle is the so-called nine-point circle. Let us examine the circumcircle of the triangle whose vertices are the midpoints of $\triangle ABC$ (the medial triangle). Firstly, what is its radius? The medial triangle is a half sized version of the original triangle (because of the midpoint theorem), so its circumradius will also be half that of the large triangle, i.e. it will be $\frac{R}{2}$.



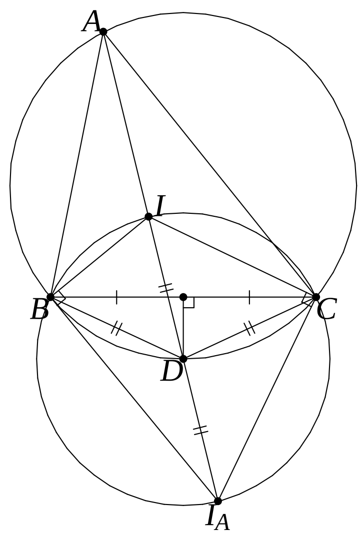
Now let us see what other points this circle passes through. From the diagram it appears that it passes through the feet of the altitudes, so let us prove this. Since F is the midpoint of the hypotenuse of $\triangle APB$, we have $\angle FPA = \angle FAP = 90^\circ - \beta$. Similarly $\angle EPA = 90^\circ - \gamma$ and so $\angle FPE = \alpha = \angle FDE$ (since $\triangle ABC \,|||\, \triangle DEF$). It follows that P lies on the circle. Similarly Q and R also lie on the circle.

Point X is the midpoint of HC, and it also appears to lie on the circle. HC is the diameter of the circle passing through H, Q, C and P, so X is the centre of this circle. It follows that $\angle PXQ = 2\angle PCQ = 2\gamma$. But $\angle PEQ = \angle PEF + \angle FEQ = \angle PDF + \angle FEQ = \gamma + \gamma$, so $\angle PEQ = \angle PXQ$ and so X lies on the circle. Similarly the midpoints of HA and HC lie on the circle.

Because there are nine well-defined points which lie on this circle, it is known as the nine-point circle.

6.6 Another circle

Consider that $\angle I_ABI = \angle I_ACI = 90^\circ$; this shows that II_A is the diameter of a circle passing through I, I_A, B and C. Where is the centre of this circle? Well, any circle passing through B and C must have its centre on the perpendicular bisector of BC, and for II_A to be the diameter, the centre must also lie on the internal bisector of $\angle A$. Hence the centre is the intersection of these two lines. As shown above, the intersection also lies on the circumcircle of $\triangle ABC$.



Exercise 6.3 ($\star$). *In acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that*

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$;

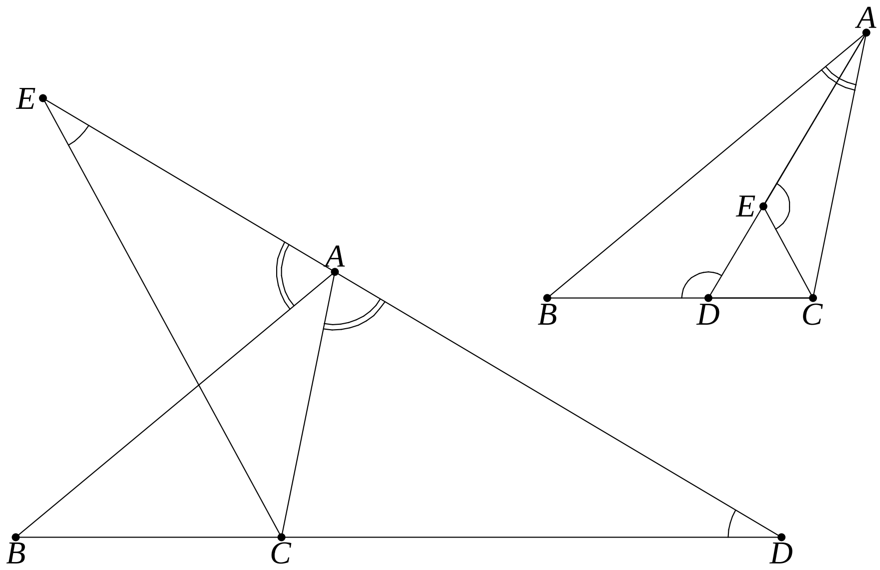
(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

6.7 Theorems

Angle bisectors can be fairly tricky to deal with. The angle bisector theorem provides a way to compute the segments which the base is divided into.

Theorem 6.1 (Angle bisector theorem). *If D is the point of intersection of BC with an angle bisector of $\angle A$, then $\frac{DB}{DC} = \frac{AB}{AC}$.*

Proof. Construct E on AD such that $\angle AEC = \angle BDA$. Then $\triangle ABD \,|||\, \triangle ACE$ (two angles) and so $\frac{DB}{EC} = \frac{AB}{AC}$. But $\triangle ECD$ is isosceles, so $CE = CD$ and therefore $\frac{DB}{DC} = \frac{AB}{AC}$ as required.



□

Exercise 6.4. *In the right-hand diagram for the angle-bisector theorem, find a formula for the length BD in terms of the side lengths a, b and c.*

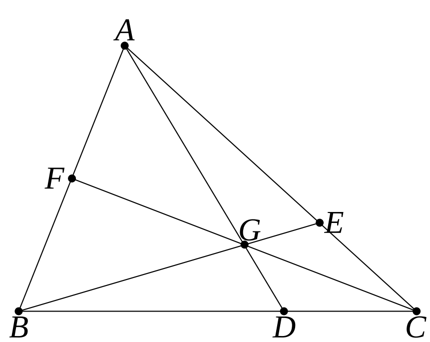
Exercise 6.5. *Given a line segment AB and a real number $r > 0$, find the locus of points P such that $\frac{AP}{BP} = r$.*

The theorems of Ceva and Menelaus are handy results when proving concurrency and collinearity respectively. They are particularly powerful because their converses are true, provided that the directions are taken into account. The converses are quite easy to prove by assuming them to be false, and then constructing two different points with the same uniquely defining properties.

Theorem 6.2 (Ceva's Theorem). *If AD, BE and CF are concurrent cevians of $\triangle ABC$ then*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Proof.



Let G be the point of concurrency.

$$\begin{aligned}
\frac{|\triangle ABD|}{|\triangle ACD|} &= \frac{BD}{DC} \quad \text{(common height)}\\
\frac{|\triangle GBD|}{|\triangle GCD|} &= \frac{BD}{DC} \quad \text{(common height)}\\
\therefore \quad \frac{|\triangle AGB|}{|\triangle CGA|} &= \frac{BD}{DC}
\end{aligned}$$

We can show similar things for $\frac{CE}{EA}$ and $\frac{AF}{FC}$. Therefore

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{|\triangle AGB|}{|\triangle CGA|} \cdot \frac{|\triangle BGC|}{|\triangle AGB|} \cdot \frac{|\triangle CGA|}{|\triangle BGC|} = 1$$

This proof has not explicitly invoked directed areas or line-segments, but if they are used it can be seen that the result will hold even if G lies outside of the triangle.

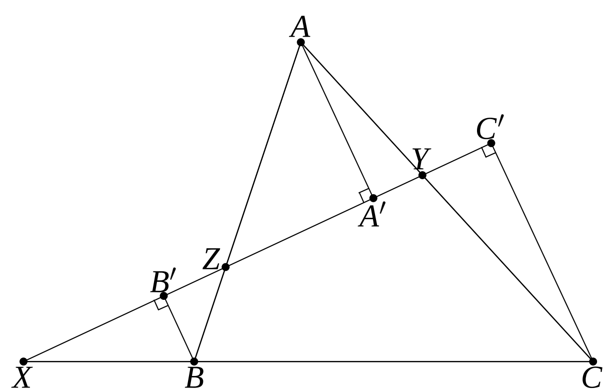
□

Theorem 6.3 (Menelaus' Theorem). *If X, Y and Z and collinear and lie on sides BC, CA and AB (or their extensions) of $\triangle ABC$ respectively, then*

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$$

(Note that the sign on the result is due to directed line segments, and indicates that the line cuts the sides themselves either twice or not at all.

Proof.



Drop perpendiculars from A, B and C to meet XYZ at A', B' and C'. From alternate angles, we have $\triangle AA'Z \,|||\, \triangle BB'Z$ and thus $\frac{AZ}{ZB} = \frac{AA'}{B'B}$. Similarly $\frac{BX}{XC} = \frac{BB'}{C'C}$ and $\frac{CY}{YA} = \frac{CC'}{A'A}$. Therefore

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{AA'}{B'B} \cdot \frac{BB'}{C'C} \cdot \frac{CC'}{A'A} = -1$$

$\square$

Exercise 6.6. *Use Menelaus' Theorem to prove Ceva's Theorem.*

Exercise 6.7 ($\star$). *ABC is an isosceles triangle with $AB = AC$. Suppose that*

(i) M is the midpoint of BC and O is the point on the line AM such that $OB \perp AB$;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

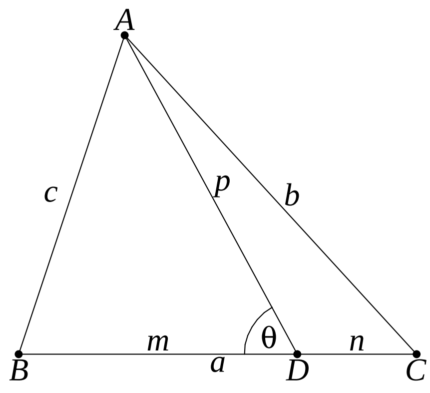
Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Stewart's Theorem is a handy tool for dealing with the length of a cevian, which is otherwise difficult to work with.

Theorem 6.4 (Stewart's Theorem). *Suppose AD is a cevian in $\triangle ABC$. Let $p = AD$, $m = BD$ and $n = CD$. Then*

$$a(p^2 + mn) = b^2 m + c^2 n.$$

Proof.



Use the cosine rule in $\triangle ABD$:

$$
\begin{aligned}
& & c^2 &= m^2 + p^2 - 2mp\cos\theta \\
&\therefore & c^2 n &= m^2 n + p^2 n - 2mnp\cos\theta
\end{aligned}
\tag{6.1}
$$

Do the same in $\triangle ACD$, noting that $\cos(180^\circ - \theta) = -\cos\theta$:

$$b^2 = n^2 + p^2 + 2np\cos\theta$$

$$\therefore \qquad b^2m = n^2m + p^2m + 2mnp\cos\theta \tag{6.2}$$

Now add (6.1) and (6.2):

$$b^2m + c^2n = m^2n + n^2m + p^2n + p^2m \tag{6.3}$$
$$= (m+n)(p^2 + mn) \tag{6.4}$$
$$= a(p^2 + mn) \tag{6.5}$$

□

In the special case that AD is a median, Stewart's Theorem reduces to $4p^2 + a^2 = 2(b^2 + c^2)$, which is known as Apollonius' Theorem.

Exercise 6.8. *In $\triangle ABC$, angle A is twice angle B. Prove that $a^2 = b(b+c)$.*

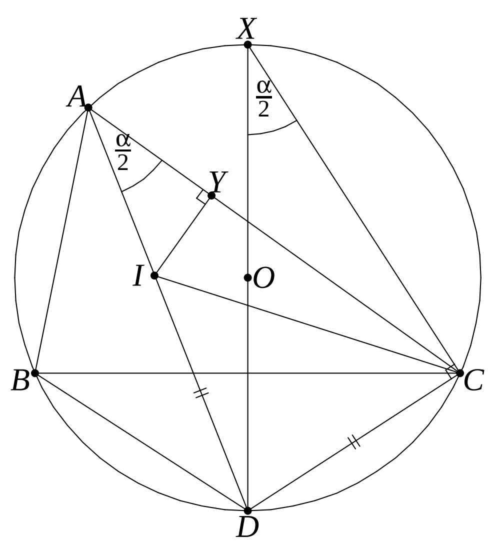
Theorem 6.5 (Euler's Formula).

$$OI^2 = R(R - 2r)$$

As a corollary, we have Euler's Inequality:

$$R \geq 2r.$$

Proof. Extend the angle bisector from A to meet the circumcircle again at D. Also construct X diametrically opposite D on the circumcircle and construct Y as the foot of the perpendicular from I onto AC. We calculate the power of I with respect to the circumcircle (see section 5.3), which is equal to $OI^2 - R^2$ and also to $-AI \cdot ID$. From section 6.6, we have $ID = CD$.



Now we note that $\triangle DXC \,|||\, \triangle IAY$, and so $\frac{AI}{IY} = \frac{XD}{DC} \iff AI \cdot ID = 2rR$. Since $OI^2 - R^2 = -AI \cdot ID$, it follows that $OI^2 = R(R-2r)$ as required. □

Euler's Theorem provides a measure of the distance between the incentre and circumcentre. However it is most often invoked as Euler's Inequality.

Exercise 6.9 ($\star$). *Let r be the inradius and R the circumradius of ABC and let p be the inradius of the orthic triangle of triangle ABC. Prove that*

$$\frac{p}{R} \leq 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2.$$

6.8 Area

There are numerous formulae for the area of a triangle, and in many cases things can be discovered by equating them.

Theorem 6.6 (Heron's Formula).

$$K = \sqrt{sxyz}$$

Proof. This is probably the ugliest proof in this booklet. Here goes:

$$\begin{aligned}
16K^2 &= 4(ab\sin\gamma)^2 \\
&= 4a^2b^2(1-\cos^2\gamma) \\
&= 4a^2b^2\left[1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \\
&= 4a^2b^2 - (a^2+b^2-c^2)^2 \\
&= (2ab - a^2 - b^2 + c^2)(2ab + a^2 + b^2 - c^2) \\
&= \left[c^2 - (a-b)^2\right]\left[(a+b)^2 - c^2\right] \\
&= (c-a+b)(c+a-b)(a+b+c)(a+b-c) \\
&= 16sxyz.
\end{aligned}$$

□

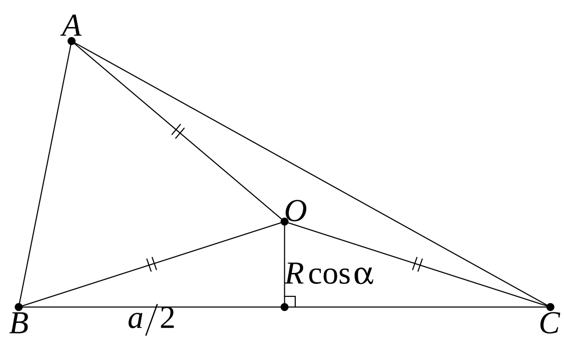
Theorem 6.7 (Triangle area formulae).

$$\begin{aligned}
K &= \tfrac{1}{2}ah_a = \tfrac{1}{2}bh_b = \tfrac{1}{2}ch_c && (6.6) \\
&= \tfrac{1}{2}ab\sin\gamma = \tfrac{1}{2}bc\sin\alpha = \tfrac{1}{2}ca\sin\beta && (6.7)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{abc}{4R} && (6.8)\\
&= 2R^2 \sin\alpha \sin\beta \sin\gamma && (6.9)\\
&= \tfrac{1}{2}R(a\cos\alpha + b\cos\beta + c\cos\gamma) && (6.10)\\
&= R(a\cos\beta\cos\gamma + b\cos\gamma\cos\alpha + c\cos\alpha\cos\beta) && (6.11)\\
&= rs && (6.12)\\
&= r_a x = r_b y = r_c z && (6.13)\\
&= \sqrt{sxyz} \qquad \textit{(Heron's Formula)} && (6.14)
\end{aligned}
$$

Proof. The first is the standard formula for the area of a triangle. The second is really the same formula, since $\sin\gamma = \frac{h_a}{b}$. The third is obtained using the extended sine rule ($\sin\gamma = \frac{c}{2R}$). The fourth is similarly obtained using the extended sine rule by converting all side lengths to sines.

Equation 6.9 is obtained by adding the areas of the isosceles triangles $\triangle BOC$, $\triangle COA$ and $\triangle AOB$. The base of $\triangle BOC$ is a and $\angle BOC = 2\angle BAC = 2\alpha$, so the height is $OC\cos\alpha = R\cos\alpha$. Adding up the areas gives the result.



The following equation is obtained from 6.9 by replacing a by $b\cos\gamma + c\cos\beta$ and similarly for b and c.

Equation 6.12 is obtained similarly to 6.9, but using I instead of O. The three triangles all have height r, so the area is $\frac{1}{2}(ra + rb + rc) = rs$. Equation 6.13 uses the excentre I_a instead; in this case one adds triangles ABI_a and ACI_a and subtracts triangle BCI_a.

Heron's Formula was covered earlier. □

Exercise 6.10. *An equilateral triangle has sides of length* $4\sqrt{3}$*. A point Q is located inside the triangle so that its perpendicular distances from two sides of the triangle are* 1 *and* 2*. What is the perpendicular distance to the third side?*

Exercise 6.11. *Prove that*

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

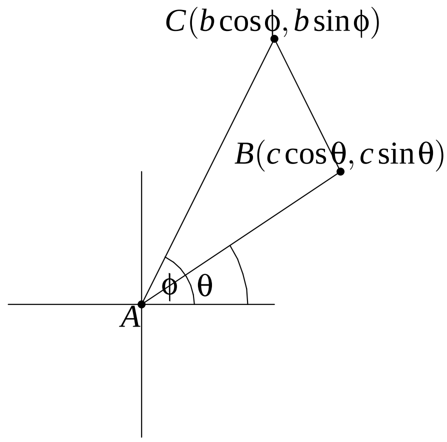
There is one area more formula that is used with coordinate geometry.

Theorem 6.8. *If one vertex of a triangle is at the origin and the other two are at* (x_1, y_1) *and* (x_2, y_2)*, then*

$$K = \frac{1}{2}\left|x_1y_2 - x_2y_1\right|.$$

If the absolute value operator is removed, one gets a formula for directed area[1].

Proof. The proof below uses trigonometry. It is also possible to compute the area of the triangle by starting with a rectangle that bounds it, and subtracting right triangles. However, that approach requires several cases to be considered.



Assume without loss of generality that C makes a larger angle from the x-axis than B (swapping B and C simply negates the term inside the absolute value). Then $(x_1, y_1) = (c\cos\theta, c\sin\theta)$, $(x_2, y_2) = (b\cos\phi, b\sin\phi)$ and the area is

$$\begin{aligned}\frac{1}{2}bc\sin\alpha &= \frac{1}{2}bc\sin(\phi - \theta)\\ &= \frac{1}{2}bc(\sin\phi\cos\theta - \cos\theta\sin\phi\\ &= \frac{1}{2}(x_1y_2 - x_2y_1).\end{aligned}$$

□

6.9 Inequalities

Inequalities in triangles are often best solved by first expressing all the quantities in terms of as few variables as possible (ideally, only two or three) and then using inequality techniques discussed in *Inequalities for the Olympiad Enthusiast* to finish the problem algebraically. Jensen's Inequality is particularly powerful when combined with trigonometric functions.

[1] The sign is used in computer graphics to determine whether three points are wound clockwise or anti-clockwise.

Theorem 6.9 (Jensen's Inequality). *A function f is said to be* convex *on an interval $[a,b]$ if $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$ for all $x,y \in [a,b]$. If f is convex*[2] *on $[a,b]$ then for any $x_1, x_2, \ldots, x_n$ in $[a,b]$ we have*

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+\cdots+f(x_n)}{n}.$$

The statement also holds if all inequality signs are reversed, in which case the function is termed concave.

Proof. Refer to page 18 of *Inequalities for the Olympiad Enthusiast*, by Graeme West. □

Exercise 6.12. *If α, β, γ are the angles of a triangle, then show that* $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}$.

One thing to keep in mind is the triangle inequality: if you reduce the problem to an inequality in a, b and c then it is possible (although not necessarily the case) that you will need to use the fact that the sum of any two is greater than the third. A technique that sometimes simplifies this to substituting $a = x+y$, $b = y+z$, $c = z+x$ in which case the triangle inequality is equivalent to $x,y,z > 0$. In some circles this has become known as the Ravi Substitution, after a Canadian IMO contestant (and later coach) Ravi Vakil. Although he did not invent the technique, he successfully applied it to an IMO problem.

There are a few other useful inequalities that are specific to triangles. The first is Euler's Inequality, mentioned above. The others are listed below.

Theorem 6.10. *In a triangle ABC,*

$$\frac{3\sqrt{3}}{2}R \geq s \qquad s^2 \geq 3\sqrt{3}K \qquad K \geq 3\sqrt{3}r^2.$$

In each case, equality occurs iff $\triangle ABC$ is equilateral.

Proof. We first prove that $\frac{3\sqrt{3}}{2}R \geq s$. From the extended sine rule, $\frac{a}{2R} = \sin\alpha$ and so

$$\begin{aligned}
\frac{s}{R} &= \sin\alpha + \sin\beta + \sin\gamma \\
&\leq 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right) \qquad \text{(Jensen's Inequality)}
\end{aligned}$$

[2] If you are familiar with calculus, a convex function is one that satisfies $f''(x) \geq 0$ for all $x \in [a,b]$.

$$
\begin{aligned}
&= 3\sin 60^\circ \\
&= \frac{3\sqrt{3}}{2}.
\end{aligned}
$$

For the remaining inequalities, we express everything in terms of x, y and z. Thus

$$
\begin{aligned}
s^2 &= s^{3/2}\sqrt{s} \\
&= \sqrt{s(x+y+z)^3} \\
&\geq \sqrt{27sxyz} \qquad \text{(AM-GM)} \\
&= 3\sqrt{3}K \qquad \text{(Heron's Formula)}. \\
K &= \frac{r^2s^2}{K} \\
&\geq \frac{3\sqrt{3}r^2K}{K} \qquad \text{(from the previous step)} \\
&= 3\sqrt{3}r^2.
\end{aligned}
$$

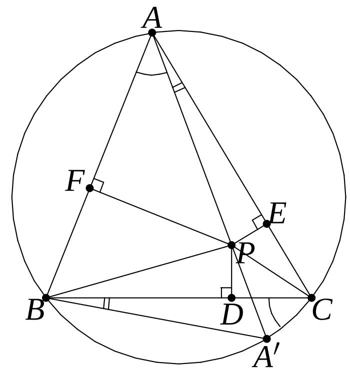
□

Theorem 6.11 (Erdős-Mordell). *Let P be a point inside triangle $\triangle ABC$, and let the feet of the perpendiculars from P to BC, CA, AB be D, E, F respectively. Then*

$$AP + BP + CP \geq 2(DP + EP + FP).$$

Proof. Extend AP to meet the circumcircle of $\triangle ABC$ at A'. Let $\angle BAP = \theta$ and $\angle CAP = \phi$. Note that $FP = AP\sin\theta$ and $EP = AP\sin\phi$, so $\frac{EP}{FP} = \frac{\sin\phi}{\sin\theta} = \frac{CA'}{BA'}$. Also note that $a \cdot AA' = b \cdot BA' + c \cdot CA'$ (from Ptolemy's Theorem in the cyclic quadrilateral $ACA'B$), so $AA' = \frac{b}{a} \cdot BA' + \frac{c}{a} \cdot CA'$. Now

$$
\begin{aligned}
AP &= \frac{FP}{\sin\theta} \\
&= \frac{FP \cdot 2R}{BA'} \qquad \text{(Extended Sine Rule)} \\
&\geq \frac{FP \cdot AA'}{BA'} \qquad (AA' \text{ is less than the diameter}) \\
&= \frac{FP(b \cdot BA' + c \cdot CA')}{a \cdot BA'} \\
&= \frac{b}{a} \cdot FP + \frac{c}{a} \cdot \frac{CA'}{BA'} \cdot FP \\
&= \frac{b}{a} \cdot FP + \frac{c}{a} \cdot EP.
\end{aligned}
$$



Now we can establish similar inequalities for BP and CP, and adding these gives

$$
\begin{aligned}
PA + PB + PC &\geq \left(\frac{b}{c} + \frac{c}{b}\right) PD + \left(\frac{c}{a} + \frac{a}{c}\right) PE + \left(\frac{a}{b} + \frac{b}{a}\right) PF \\
&\geq 2(PD + PE + PF). \qquad \text{(AM-GM)}
\end{aligned}
$$

□

Exercise 6.13. *Let ABC be a triangle and P be an interior point in ABC. Show that at least one of the angles PAB, PBC, PCA is less than or equal to 30 degrees.*

7 Transformations

A very powerful idea in geometry is that of a transformation. A transformation maps every point in space to some other point in space. Structures like lines or circles are transformed by applying the transformation to every point on them. They do not necessary maintain their shapes; in fact there is a transformation (inversion) which generally maps lines to circles! Each transformation will preserve certain properties of a diagram, and by translating the properties of the original into the transformed diagram one can obtain new information. Here a diagram is really just a set of points.

7.1 Affine transformations

The transformations we discuss here are all *affine*. That means that straight lines are mapped to straight lines, and lengths are scaled uniformly. The transformations presented here all preserve angles as well. These transformations can in fact be built up by combining reflections and scale changes, although this is not necessarily the best way to think about them.

7.2 Translations, rotations and reflections

The simplest transformation is a translation: every point simply moves a constant distance in a constant direction; this is like picking up a piece of paper and moving it, without rotating it. Rotations rotate all the points by some angle around a particular point; this is like sticking a pin in a piece of paper and then turning it. Reflections take all points and reflect them in a particular line; this is like picking up the piece of paper and putting it down upside-down (the paper would of course need to be thin enough for the diagram to be seen through the back).

While these are all quite straightforward, they can also be very powerful because they preserve so much. They are also closely related, as shown by the next problem.

Exercise 7.1. *In each of the following, show that the transformations exist using a concrete construction.*

(a) Show that any rotation or translation can be expressed as the combination of a pair of reflections, or vice versa.

(b) Show that two rotations, two translations or a translation and rotation can always be combined to produce a single translation or rotation.

(c) Show that any combination of translations, reflections and rotations yields either a rotation, a translation, or a translation followed by a reflection.

Exercise 7.2. *In acute-angled triangle ABC, a point P is given on side BC. Show how to find Q on CA and R on AB such that $\triangle PQR$ has the minimum perimeter.*

Exercise 7.3 ($\star$). *The point O is situated inside the parallelogram $ABCD$ so that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$.*

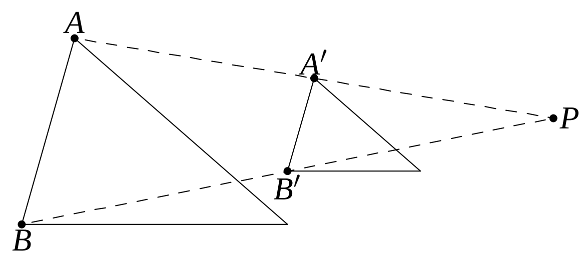
7.3 Homothetisms

So far we have discussed only *rigid* transforms, namely those that can be illustrated with a piece of paper. We now move on to scaling. Imagine drawing a diagram on a new T-shirt, and then letting the T-shirt shrink in the wash. Assume the ink doesn't run and that the T-shirt doesn't warp, you will have the same diagram, only smaller. All the angles and so on will be the same, although lengths will not.

A *homothetism* is a fancy name for scaling. One chooses a centre (sometimes called the "centre of similitude") and a scale factor. Every point is then kept in the same direction relative to the centre, but its distance from the centre is scaled by the scale factor. Like translations, homothetisms preserve orientation, angles, and ratios of lengths. However, lengths are scaled by the scale factor. The result below allows one to find the centre of a homothetism.

Theorem 7.1. *Let S and T be two similar figures which have the same orientation, but are not the same size. Then there is a homothetism that maps S to T.*

Proof. Pick a point A in S and its corresponding point A' in T. Now pick a second point B in S, not on AA', and its corresponding point in B[3]. Now if AA' and BB' are parallel then $AA'B'B$ would be a parallelogram, making $AB = A'B'$. But we assumed that S and T are of different sizes, which would give a contradiction. Hence AA' and BB' meet at a point, which we will call P. Now consider the homothetism with centre of similitude P and scale factor $\frac{A'P}{AP}$. It will clearly map A to A'; will it map B to B'? Yes, because $\triangle ABP \,|||\, \triangle A'B'P$ by parallel lines. If we can show that this homothetism maps the rest of S to T then we are done.



Let C be some arbitrary point in S. We aim to show that the homothetism maps C to its corresponding point C' in T. If C is A or B then we are done. If C lies on AB then C is uniquely defined by $\frac{AC}{BC}$ (with directed line segments). But homothetisms preserve ratios of lengths, and $\frac{A'C'}{B'C'} = \frac{AC}{BC}$ so C is mapped to C'. If C does not lie on AB then C is uniquely defined by the directed angles $\angle BAC$ and $\angle ABC$, and angles are preserved by homothetisms. $\square$

The construction also suggests how the centre of similitude can be found in practice: take two pairs of corresponding points and find the intersection of the lines between them. For example, any two circles of different sizes satisfy the requirements, so a homothetism can be found between them. The points of tangency of the common tangent are corresponding points, since they have the same orientation relative to the centre. Hence the centre of similitude is the intersection of the common tangents.

What happens if we have non-overlapping circles, and use the *other* pair of common tangents? It turns out that this point is also a centre of similitude. However, this homothetism has a negative scale factor, which means that points are "sucked"

[3] If no such B exists, then make some arbitrary construction in S and the corresponding construction in T to produce such a B.

through the centre and pushed out the other side. This also rotates the figure by 180°, although for a circle this isn't visible. The theorem above in fact applies to situations where the two figures have orientations that are out by 180°, in which case a negative scale factor is used. In this case the figures may even by the same size (since the scale factor is -1, not 1).

Exercise 7.4. *Let ABC be a triangle. Use a homothetism to show that*

(a) the medians of $\triangle ABC$ are concurrent;

(b) the point of concurrency (the centroid) divides the medians in a $2:1$ ratio;

(c) the orthocentre H, the centroid G and the circumcentre O are collinear, with $HG:GO = 2:1$ (this line is known as the Euler line). Assume that H and O exist (i.e. that the defining lines are concurrent).

Exercise 7.5 ($\star$). *On a plane let C be a circle, L be a line tangent to the circle C and M be a point on L. Find the locus of all points P with the following property: there exist two points Q,R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.*

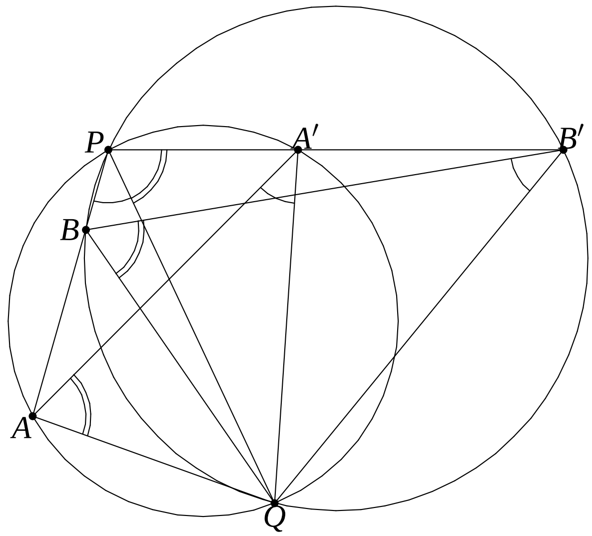
7.4 Spiral similarities

An even more general transformation than a homothetism is a spiral similarity. A spiral similarity combines the effects of a homothetism and a rotation: the plane is not only scaled around a centre P by some factor r, it is also rotated around P by an angle θ. A spiral similarity preserves pretty much the same things as homothetisms i.e. ratio of lengths and angles. However, corresponding lines are no longer parallel, but meet each other at an angle of θ. As for homothetisms, there is a result that makes it possible to find a spiral similarity given two similar figures.

Theorem 7.2. *Let S and T be two sets of points that are similar but have either different orientation or different size (or both). Then there is a spiral similarity that maps S to T.*

Proof. In the special case that S and T have the same orientation, there exists a homothetism, which is just a special case of a spiral similarity. So we assume that S and T have different orientations. We also include the case where S and T are oriented 180° apart in the special case, as this is a homothetism with negative scale factor.

Choose two arbitrary points A and B in S, and their corresponding points A' and B' in T. Let P be the intersection of AB and $A'B'$. Construct the circumcircles of $\triangle AA'P$ and $\triangle BB'P$, and let their second point of intersection be Q (Q exists because of the assumptions).



Now $\angle AQA' \equiv \angle APA' \equiv BPB' \equiv BQB'$, $\angle AA'Q \equiv \angle APQ \equiv \angle BPQ \equiv \angle BB'Q$ and similarly $\angle A'AQ \equiv B'BQ$. It follows that triangles $AA'Q$ and $BB'Q$ are directly similar[4]. Now consider the spiral similarity with centre Q, angle AQA' and scale factor $\frac{A'Q}{AQ}$. It will map A to A' by construction, and from the similar triangles it will map B to B'. We can now proceed to show that S is mapped to T, as was done in the corresponding theorem for homothetisms. $\square$

Exercise 7.6. *Squares are constructed outwards on the sides of triangle ABC. Let P, Q and R be the centres of the squares opposite A, B and C respectively. Prove that AP and QR are equal and perpendicular.*

8 Miscellaneous problems

These problems all draw on the techniques in this book, but do not fit well into any particular section. They are mostly very challenging problems designed to give you practice.

Exercise 8.1 ($\star$). *ABCD is a square. P is a point inside the square with* $\angle ABP = \angle BAP = 15^\circ$. *Show that* $\triangle CDP$ *is equilateral.*

Exercise 8.2 ($\star$). *A 6m tall statue stands on a pedestal, so that the foot of the statue is 2m above your head height. Determine how far from the statue you should stand so that it appears as large as possible in your vision.*[5]

[4] Two triangles are directly similar if they are similar and have the same clockwise/anti-clockwise orientation.

[5] In other words, maximise the angle formed by the foot of the statue, your head and the top of the statue.

Exercise 8.3 ($\star$). *In an acute angled triangle ABC the interior bisector of $\angle A$ intersects BC at L and the circumcircle of $\triangle ABC$ again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.*

Exercise 8.4 ($\star$). *ABC is a triangle. The internal bisector of the angle A meets the circumcircle again at P. Q and R are similarly defined relative to B and C. Prove that*

$$AP + BQ + CR > AB + BC + CA.$$

Exercise 8.5 ($\star$). *A circle of radius r is inscribed in a triangle ABC with area K. The points of tangency with BC, CA and AC are X, Y and Z respectively. AX intersects the circle again in X'. Prove that $BC \cdot AX \cdot XX' = 4rK$.*

Exercise 8.6 ($\star$). *A semicircle is drawn on one side of a straight line ℓ. C and D are points on the semicircle. The tangents at C and D meet ℓ again at B and A respectively, with the centre of the semicircle between them. Let E be the point of intersection of AC and BD, and F the point on ℓ such that EF is perpendicular to ℓ. Prove that EF bisects $\angle CFD$.*

Exercise 8.7 ($\star$). *In $\triangle ABC$, let D and E be points on the side BC such that $\angle BAD = \angle CAE$. If M and N are, respectively, the points of tangency with BC of the incircles of $\triangle ABD$ and $\triangle ACE$, show that $\dfrac{1}{MB} + \dfrac{1}{MD} = \dfrac{1}{NC} + \dfrac{1}{NE}$.*

Exercise 8.8 ($\star$). *Let P be a point inside $\triangle ABC$ such that*

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incentres of $\triangle APB, \triangle APC$ respectively. Show that AP, BD and CE meet at a point.

9 Solutions

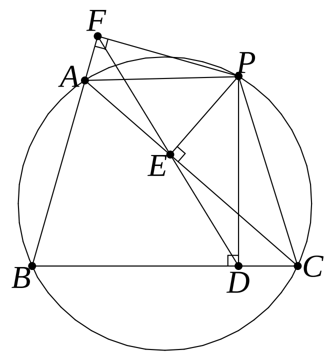
3.1 Using classical geometry to solve this problem would result in an enormous number of different cases. However, directed angles hide all of that, and the result appears with a few lines of basic calculation:

$$\begin{aligned}
\angle AZC &\equiv \angle AZO + \angle OZC \\
&\equiv \angle AXO + \angle OYC && \text{(concyclic points)} \\
&\equiv \angle BXO + \angle OYB && \text{(collinear points)}
\end{aligned}$$

$$
\begin{aligned}
&\equiv \angle BXO + \angle OXB && \text{(concyclic points)}\\
&\equiv \angle BXO - \angle BXO && \text{(directed angles)}\\
&\equiv 0^\circ,
\end{aligned}
$$

and hence A, Z and C are collinear.

3.2 Note that PC subtends right angles at D and E, and hence is the diameter of a circle passing through P, C, D and E. Similarly, P, A, F and E are concyclic.



$$
\begin{aligned}
\angle DEF &\equiv \angle DEP + \angle PEF\\
&\equiv \angle DCP + \angle PAF\\
&\equiv \angle BCP - \angle BAP.
\end{aligned}
$$

It follows that $\angle DEF \equiv 0^\circ \iff \angle BCP \equiv \angle BAP$. The first is a condition for D, E, F to be collinear and the second is a condition for P to lie on the circumcircle of $\triangle ABC$.

4.1 $\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot A \pm \cot B}$ can be shown by substituting $\tan\theta = \frac{1}{\cos\theta}$ into $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ and simplifying. The expressions for $\sin A \sin B$ and similar expressions can be proved simply by expanding the right hand side and cancelling terms. The final three equations are derived by making suitable substitutions into the previous three.

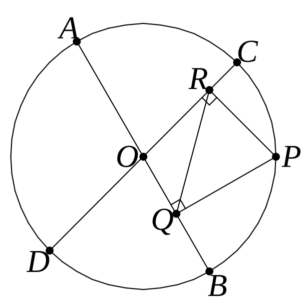
4.2 We first derive a general formula for $\tan(A+B+C)$.

$$
\begin{aligned}
\tan(A+B+C) &= \tan[(A+B)+C]\\
&= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}\\
&= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \cdot \tan C}
\end{aligned}
$$

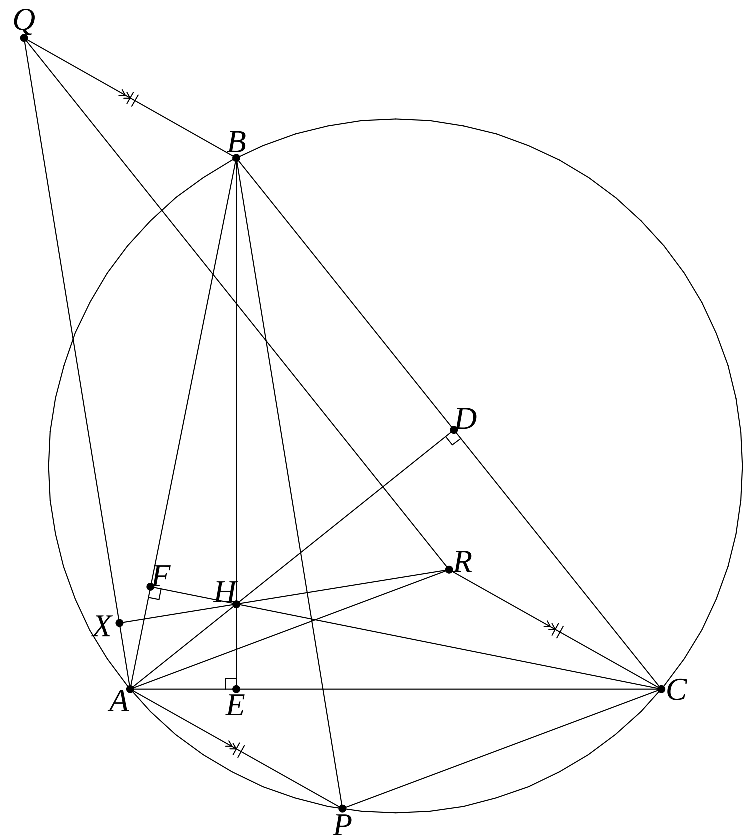
$$= \frac{\tan A + \tan B + (1 - \tan A \tan B)\tan C}{(1 - \tan A \tan B) - (\tan A + \tan B)\tan C}$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

However, we know that $\tan(A+B+C) = \tan 180^\circ = 0$, so the numerator must be 0. The result follows.

4.3 Suppose that P lies on the arc BC, as in the diagram. Then $OQPR$ is cyclic with diameter OP, so applying the extended sine rule in $\triangle OQR$ gives $QR = OP \sin\angle BOC$. Now $\angle BOC$ is fixed and OP is the radius of the circle, also fixed. So QR is fixed if P lies on the arc BC. But $\sin\angle BOC = \sin\angle COA = \sin\angle DOA = \sin\angle DOB$, so QR is constant wherever on the circle P may be.



5.1 This problem is fairly straight-forward as it consists almost entirely of angle chasing. The only difficulty is that P can lie anywhere on the circumcircle, which could give rise to multiple cases. We can get around this with directed angles. This diagram is thus only for reference. D and F are the feet of the altitudes from A and C in $\triangle ABC$.



(a) Firstly notice that since $PAQB$ and $PARC$ are parallelograms, BQ and CR are parallel and equal (and in the same direction), so $BCRQ$ is also a parallelogram. It follows that $RQ \parallel CB$ and hence $AH \perp RQ$. This shows that H lies on one altitude of $\triangle AQR$. If $RX \perp AQ$ then it would lie on another altitude we would be done.

Note that B, D, H and F are concyclic. Thus

$$\begin{aligned}
\angle AHC &\equiv \angle DHF && \text{(opposite angles)}\\
&\equiv \angle DBF && (D, H, F, B \text{ concyclic})\\
&\equiv \angle CBA && \\
&\equiv \angle CPA && (A, B, C, P \text{ concyclic})\\
&\equiv \angle ARC && (AP \parallel RC, AR \parallel PC)
\end{aligned}$$

and therefore A, H, R and C are concyclic. Thus

$$\begin{aligned}
\angle AXR &\equiv \angle XAR + \angle ARX\\
&\equiv \angle XAB + \angle BAC + \angle CAR + \angle ARH\\
&\equiv \angle QAB + \angle FAC + \angle ACP + \angle ACH\\
&\equiv \angle PBA + \angle ABP + \angle FAC + \angle ACF
\end{aligned}$$

$$\begin{aligned} &\equiv \angle AFC \\ &\equiv 90^\circ \end{aligned}$$

and the result follows.

(b) This is just more angle chasing, using the fact that H, X, A and E are concyclic (because of the right angles).

$$\begin{aligned} \angle AEX &\equiv \angle AHX \\ &\equiv \angle AHR \\ &\equiv \angle ACR \\ &\equiv \angle PAC \\ &\equiv \angle PAE \end{aligned}$$

from which it follows that $XE \parallel AP$.

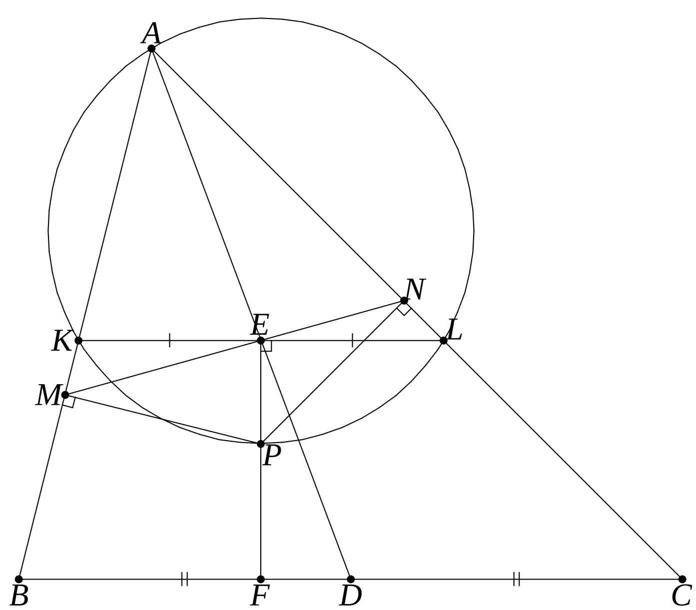
(Proposed for IMO 1996)

5.2 We use Ptolemy's Inequality:

$$\begin{aligned} & AP \cdot BC + BP \cdot CA \geq CP \cdot AB \\ \iff \quad & AP + BP \geq CP \qquad \text{(since } AP = BP = CP\text{).} \end{aligned}$$

Equality occurs if and only if $ABPC$ is a cyclic quadrilateral.

5.3 Construct KL through E parallel to BC, with K and L on AB and AC respectively.



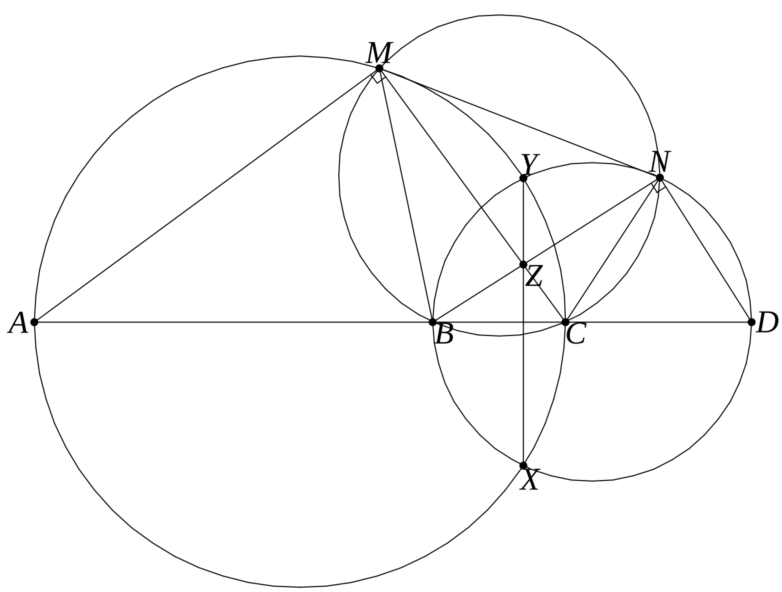
From similar triangles AKE and ABD, we have $KE = BD \cdot \frac{AE}{AD}$. Similarly, $EL = DC \cdot \frac{AE}{AD}$. But $BD = DC$, so $KE = EL$ and hence AE is a median of $\triangle AKL$. Also, $PE \perp KL$ (since $KL \parallel BC$), so M, E and N are the pedal points of P in triangle AKL. The Simpson Line theorem states that M, E and N are collinear if and only if P lies on the circumcircle of $\triangle AKL$. But the perpendicular bisector of KL and the angle bisector of $\angle A$ both meet the circumcircle at the middle of the arc KL, so P lies on the angle bisector of $\angle A$.

(*Crux Mathematicorum*, 1990, 293)

5.4 If P is one of the midpoints, then the lengths of the tangents from P to the two circles are equal. Since these lengths are the square roots of the power of P with respect to these two circles, P must lie on the radical axis. Since this is true for four midpoints, they are collinear because the radical axis is a straight line.

5.5 Call the given circles Γ_1 and Γ_2, and construct a third circle Γ_3 which intersects both Γ_1 and Γ_2. The position of Γ_3 is arbitrary, provided that the centres of the three circles are not collinear. The radical axes of (Γ_1, Γ_2) and (Γ_1, Γ_3) can be found by drawing lines through the intersection points. The intersection of these two lines is the radical centre of the three circles. The desired radical axis now passes through the radical centre and is perpendicular to the line of centres of Γ_1 and Γ_2, which can easily be constructed.

5.6 We use directed angles and line segments, since P may lie either inside or outside of the segment XY. It is also possible (but more tedious) to do the proof with two cases. The diagram below shows the one case.



Label the circle with diameter AC as Γ_1, and the circle with diameter BD as Γ_2. The point Z lies on the radical axis of the two circles, so it has equal power with respect to both. In particular, $ZM \cdot ZC = ZN \cdot ZB$, which prove that M, N, B and C are concyclic. Call this circle Γ_3. Now

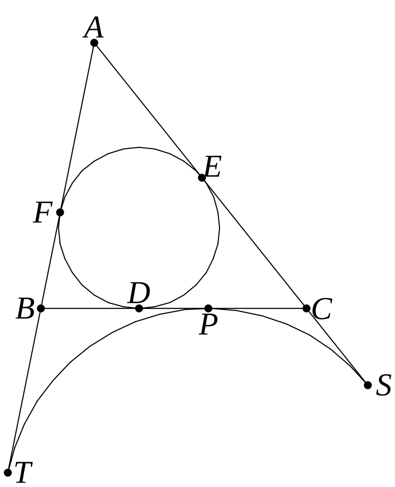
$$\begin{aligned}\angle MND &\equiv \angle MNB + \angle BND \\ &\equiv \angle MCB + 90^\circ \\ &\equiv \angle MCA + \angle AMC \\ &\equiv -\angle CAM \\ &\equiv \angle MAD.\end{aligned}$$

This proves that M, N, A and D are also concyclic; call this circle Γ_4. Finally, we note that AM, DN and XY are the three radical axes formed between the circles Γ_1, Γ_2 and Γ_4. These lines are not all parallel ($AM \parallel XY$ would require that $P = Z$), so they must coincide at the radical centre of the circles.

(IMO 1995, problem 1)

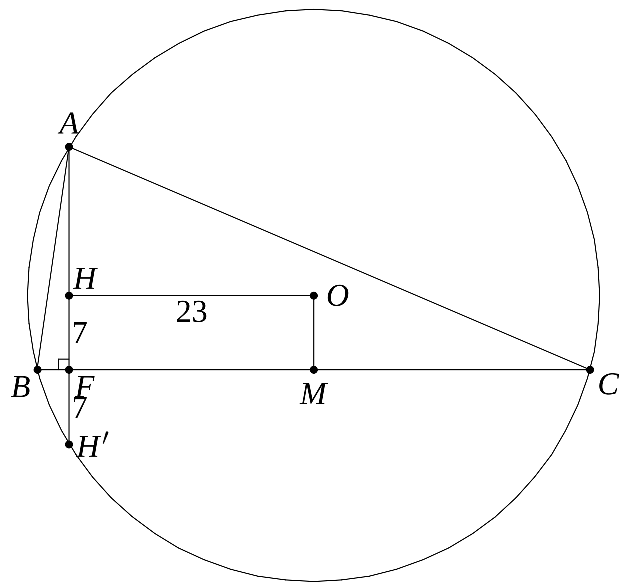
6.1 Let D, E and F be the points of tangency of the incircle with BC, CA, AB and let the excircle be tangent to the same sides at P, S and T respectively. Then from common tangents,

$$\begin{aligned}2ES = 2FT &= ES + FT \\ &= EC + CS + FB + BT \\ &= DC + CP + DB + BP \\ &= 2BC.\end{aligned}$$



Hence $ES = FT = BC = y + z$. Now $BP = BT = FT - BF = (y+z) - y = z$. Similarly, $CP = y$.

6.2 Since the altitude AF passes through H and $BC \perp AF$, it follows that BC and FM coincide. Let H' be the reflection of H in BC. H' is known to lie on the circumcircle of $\triangle ABC$, so $R = H'O = \sqrt{23^2 + 14^2}$. Hence $BM = \sqrt{H'O^2 - 7^2} = 26$ and $BC = 2BM = 52$.



6.3 Clearly, A_0, B_0 and C_0 are in fact I_A, I_B and I_C, and we will refer to them as such.

(i) We will show that $|\triangle II_AC| = 2|\triangle IA_1C|$ (refer to the diagram on page 15, where D is A_1). Results for five other pairs of triangles follow similarly, and adding them all up gives the desired result. Triangles II_AC and IA_1C have a common height, and bases II_A and IA_1. But these bases are the radius and diameter of the circle with diameter II_A, so the result follows.

(ii) It suffices to show that $|AC_1BA_1CB_1|$ is at least twice $|ABC|$, which is equivalent to showing that $|\triangle BCA_1| + |\triangle CAB_1| + |\triangle ABC_1| \geq |\triangle ABC|$. Let A_2, B_2 and C_2 be the reflections of H in BC, CA and AB. These points are known to lie on the circumcircle. When comparing the areas of triangles BCA_1 and BCA_2, we note that they share a common base but the height of $\triangle BCA_1$ is greater than or equal to that of $\triangle BCA_2$. Hence

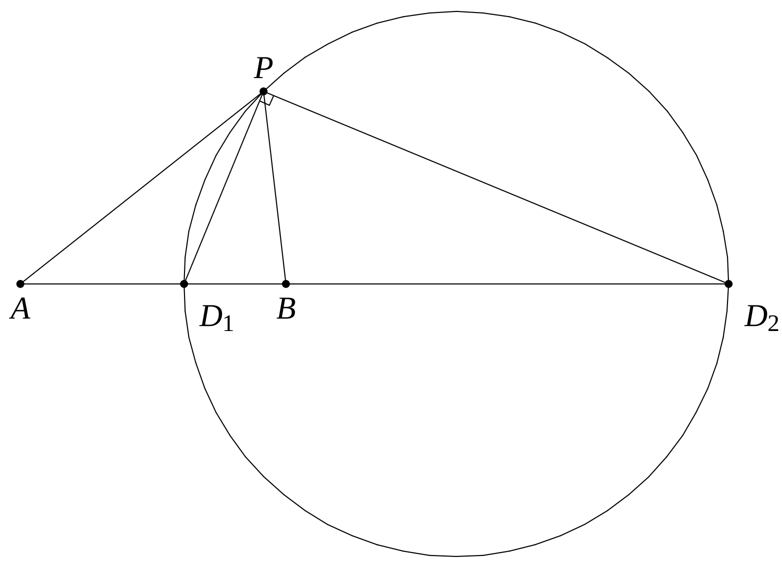
$$\begin{aligned} |\triangle BCA_1| + |\triangle CAB_1| + |\triangle ABC_1| &\geq |\triangle BCA_2| + |\triangle CAB_2| + |\triangle ABC_2| \\ &= |\triangle BCH| + |\triangle CAH| + |\triangle ABH| \\ &= |\triangle ABC|. \end{aligned}$$

(IMO 1989 Question 2)

6.4 Let $BD = m$ and $DC = n$. Then $m+n = a$ and $\frac{n}{m} = \frac{a-m}{m} = \frac{b}{c}$. Hence

$$BD = m = \frac{a}{1+\frac{b}{c}} = \frac{ac}{b+c}.$$

6.5 If $r = 1$, then $AP = BP$ and so the locus is simply the perpendicular bisector. Otherwise suppose $r > 1$ (the situation is symmetric if $r < 1$). Pick an arbitrary P not on AB which satisfies the condition. Let the internal and external angle bisectors of $\angle APB$ meet AB at D_1 and D_2 respectively. Then by the angle bisector theorem, $\frac{AD_1}{BD_1} = \frac{AD_2}{BD_2} = r$. D_1 and D_2 are the only two points on AB that satisfy this, so they are fixed independent of P. Also, $\angle D_1PD_2 = 90°$, so P must lie on the circle with diameter D_1D_2.

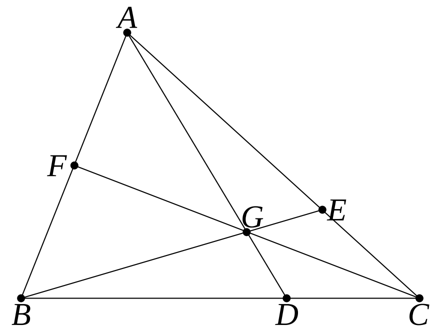


Conversely, suppose P lies on this circle. If P also lies on AB then $P = D_1$ or $P = D_2$, both of which satisfy the conditions. Otherwise let the internal and external bisectors of $\angle PAB$ meet AB at E_1 and E_2 respectively. If $\frac{AP}{BP} = \frac{AE_1}{BE_1} = \frac{AE_2}{BE_2} < r$ then E_1 lies closer to A than D_1 and E_2 lies further from A than D_2. But this means that $\angle E_1PE_2 > 90°$, which is a contradiction. Similarly, if $\frac{AP}{BP} > r$ then $\angle E_1PE_2 < 90°$, again a contradiction. Thus $\frac{AP}{BP} = r$, and this circle is precisely the locus of P.

This circle is known as an *Apollonius* circle.

6.6 Apply Menelaus to $\triangle ACD$ cut by line BGE:

$$\frac{AG}{GD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = -1. \tag{9.1}$$

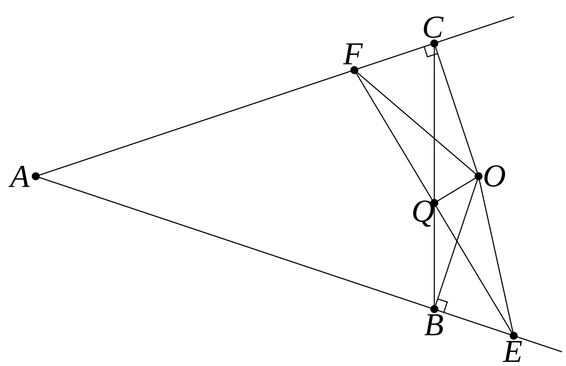


Similarly, one can apply it to $\triangle ABD$ cut by BGE:

$$\frac{AG}{GD} \cdot \frac{DC}{CB} \cdot \frac{BF}{FA} = -1. \tag{9.2}$$

Finally, dividing (9.1) by (9.2) and doing some re-arranging (while being careful with the sign conventions) gives Ceva's Theorem.

6.7 Without loss of generality, let $BQ \leq CQ$, giving the diagram below:



Suppose $OQ \perp EF$. Then $EBQO$ and $FCOQ$ are cyclic quadrilaterals, so $\angle BEO = 180^\circ - \angle BQO = \angle CQO = \angle CFO$. But $BO = CO$, so $\triangle BEO \equiv \triangle CFO$. This gives $EO = FO$, making $\triangle EOF$ isosceles. But $OQ \perp EF$, so $EQ = QF$.

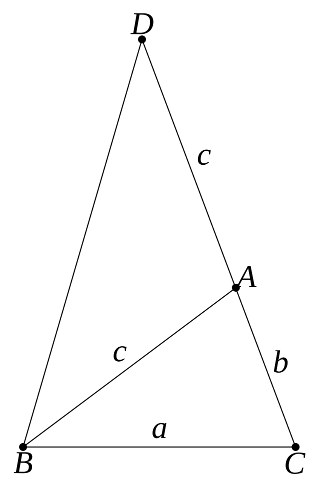
Now suppose that $QE = QF$. Apply Menelaus to triangle AEF, cut by line BQC:

$$1 = \frac{EQ}{QF} \cdot \frac{FC}{CA} \cdot \frac{AB}{BE} = \frac{FC}{BE}.$$

Hence $CF = BE$. Also, $BO = CO$, so $\triangle BEO \equiv \triangle CFO$ and hence $EO = FO$. Then $\triangle EOF$ is isosceles with $EQ = QF$, so $OQ \perp EF$.

(IMO 1994 question 2)

6.8 Construct D on the extension of AC such that $\angle ABD = \angle ABC$. Note that AB is then an angle bisector of $\triangle BDC$. Also, $\angle BDA = 2\angle ABC - \angle ABD = \angle ABD$, so triangle ABD is isosceles and $AD = c$. From the angle bisector theorem (or from $\triangle ABC \,|||\, \triangle BDC$), we find that $AD = \frac{ac}{b}$.



From Stewart's Theorem, we get

$$
\begin{aligned}
& (b+c)(c^2+bc) = \left(\frac{ac}{b}\right)^2 \cdot b + a^2 c \\
\implies \quad & (b+c)^2 bc = a^2c^2 + a^2bc \\
\implies \quad & b(b+c) = a^2,
\end{aligned}
$$

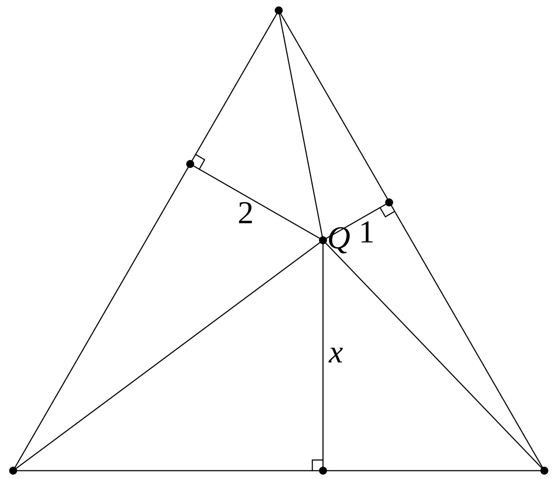
as required.

6.9 Let the orthic triangle be $A'B'C'$. We use Euler's Inequality twice, once on $\triangle ABC$ and once on $\triangle A'B'C'$. The vertices of the orthic triangle lie on the nine-point circle, so the circumradius of $\triangle A'B'C'$ is $R/2$. Thus

$$
\begin{aligned}
\frac{p}{R} &= \frac{1}{2} \cdot \frac{p}{R/2} \\
&\le \frac{1}{4} \\
&= 1 - \frac{1}{3} \cdot \frac{3}{2}^2 \\
&\le 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2.
\end{aligned}
$$

(Proposed at IMO 1993)

6.10 The height of the triangle is 6, so the area is $12\sqrt{3}$. Let the required length be x, and consider the area as the sum of the areas of the triangles formed by Q and the vertices.



The total area is thus $2\sqrt{3}(1+2+x)$. Solving the equation $12\sqrt{3} = 2\sqrt{3}(1+2+x)$ gives $x = 3$.

6.11 We know that $s = x+y+z$. Divide through by K, recalling that $K = rs = r_a x = r_b y = r_c z$.

6.12 We first check that sin is concave on $[0^\circ, 180^\circ]$:

$$\frac{\sin x + \sin y}{2} = \sin\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right) \leq \sin\left(\frac{x+y}{2}\right).$$

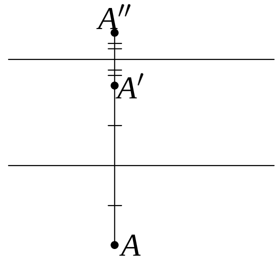
Thus

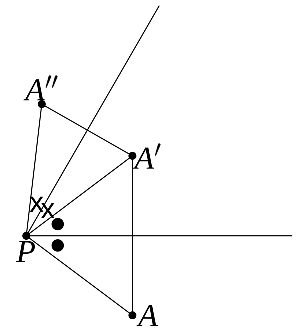
$$\sin\alpha + \sin\beta + \sin\gamma \leq 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3\sin 60^\circ = \frac{3\sqrt{3}}{2}.$$

6.13 Suppose for a contradiction that these angles are all strictly greater than 30°. Drop perpendiculars from P onto BC, CA, AB to meet at D, E, F respectively. Then $2PF > PA$, $2PD > PB$ and $2PE > PC$. But then $PA + PB + PC < 2(PD + PE + PF)$, which contradicts the Erdős-Mordell Theorem.

(IMO 1991, question 5)

7.1 (a) When combining two reflections, there are two cases.





In the diagrams above, the first reflection maps A to A', and the second maps A' to A''.

(i) The lines of reflection are parallel, separated by a distance d. As can be seen from the diagram, the combination of the reflections is a translation by $2d$, perpendicular to the lines of reflection (the direction depends on the order in which the reflections are performed. Conversely, any translation can be expressed as the combination of two parallel reflections, suitably oriented, and with separation equal to half the distance of the translation.

(ii) The lines of reflection are not parallel, and intersect at some point P with an angle of θ. From the diagram, it is now clear that any other point is rotated by an angle of 2θ around P, with the direction depending on the order of the rotations. Conversely, any rotation can be expressed as the combination of two reflections which pass through the centre of the rotation, and with an angle between them of half the rotation angle.

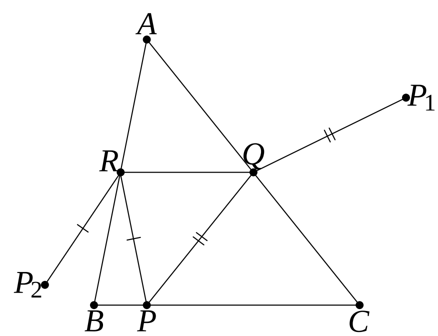
(b) Two translations trivially produce another translation, whose displacement is the vector sum of the original displacements. When one or both of the transformations is a rotation, express the transformations as pairs of reflections. We showed in part (a) that there is some freedom in the choice of reflections. We will have four reflections which are applied in order, say $b_2b_1a_2a_1$.[6] We can always choose the reflections such that a_2 and b_1 are the same. Identical reflections cancel out, so we are left with a_1b_2 which from (a) is equivalent to a rotation or translation.

(c) We can transform all the rotations and translations into pairs of reflections, using part (a). We can then pair off these reflections and convert them back into translations and rotations, possibly leaving one reflection at the end. Now part (b) shows that we can reduce the sequence of translations and

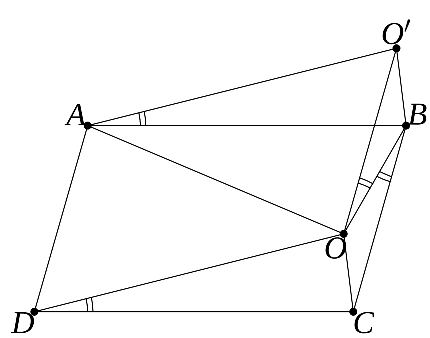
[6] We write sequence of transformations from right to left. This is because they are functions, so applying ab to a point P actually means $a(b(P))$, with b being applied fi rst.

rotations to just one, which may be followed by a reflection. It remains to show that a rotation followed by a reflection is equivalent to a translation followed by a reflection. We do this by appending two identical (and hence cancelling) reflections to the sequence, at an angle we will choose in a moment. The sequence will now appear as $ccb(r_2r_1)$ where r_2r_1 is the rotation, and c is the newly added reflection. We choose c so that cb forms a rotation with angle exactly opposite to the angle of r_2r_1. Now $(cb)(r_2r_1)$ is the combination of two rotations that forms some translation, say T (it is a translation, not a rotation, because of the choice of angle). Thus the entire sequence is equivalent to cT i.e. a translation followed by a reflection.

7.2 Reflect P in CA to obtain P_1 and reflect P in AB to obtain P_2. Now $PQ+QR+RP = P_1Q+QR+RP_2$. This sum will clearly be smallest when P_1, Q, R and P_2 lie in a straight line. So choose Q and R to be the intersections of P_1P_2 with CA and AB.



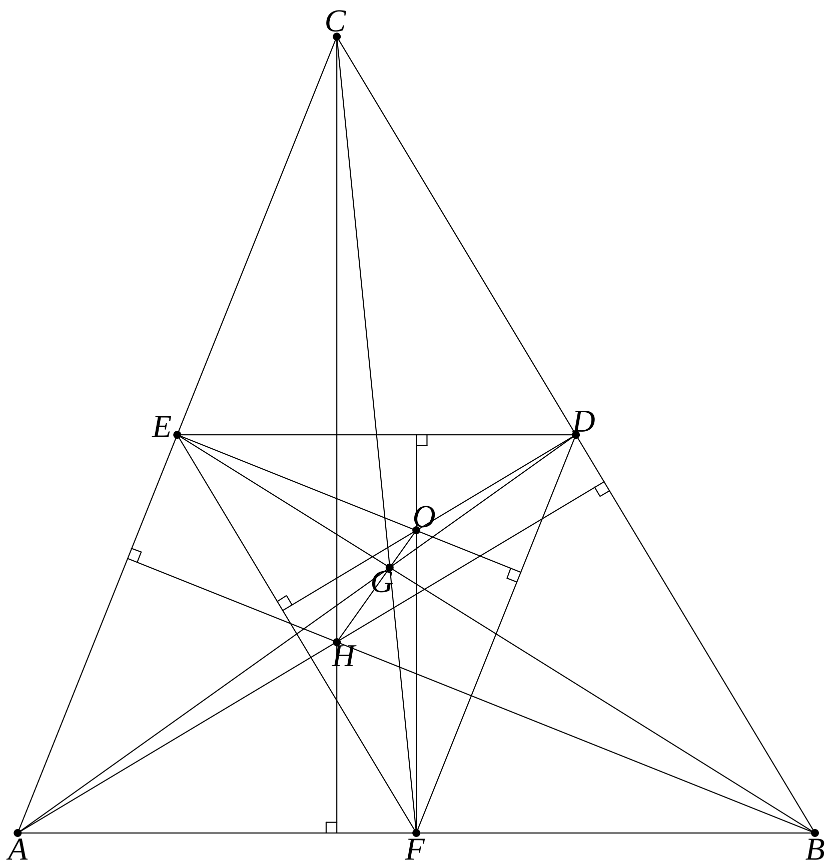
7.3 Having two supplementary angles vertically opposite each other is not very helpful. It would be more useful if we could get the angles to be either adjacent (to create a straight line) or opposite angles of a quadrilateral (to make it cyclic). One way to do this is to "pick up" triangle DOC and place DC on top of AB.



More formally, construct O' outside $ABCD$ such that $\triangle AO'B \equiv \triangle DOC$. Then $\angle AO'B + \angle AOB = 180^\circ$, so $AO'BO$ is cyclic. Also, $OO'BC$ is a parallelogram because $O'B$ and OC are equal and parallel. Thus $\angle OBC = \angle BOO' = \angle BAO' = \angle ODC$.

(Canadian Mathematical Olympiad 1997)

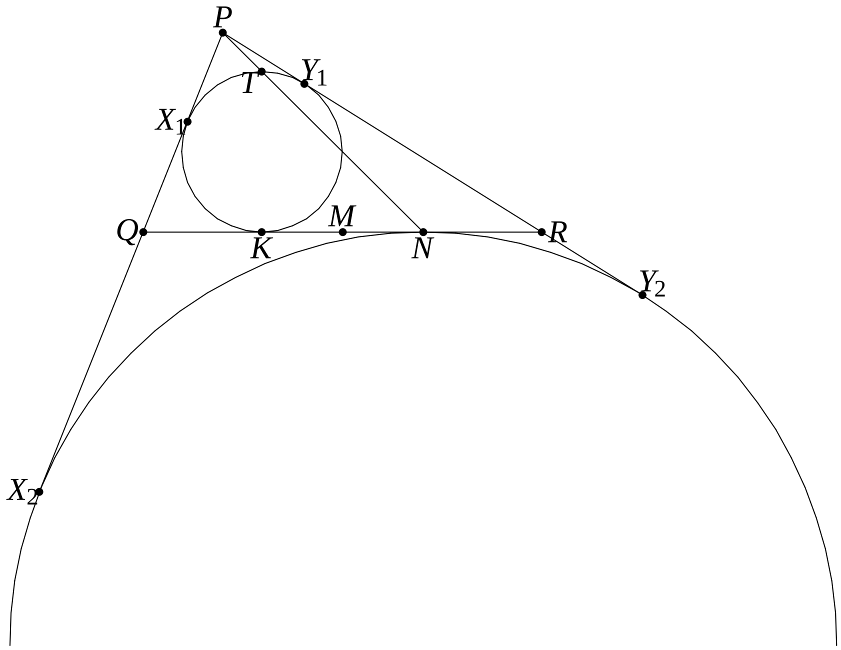
7.4 (a) Let D, E and F be the midpoints of BC, CA and AB respectively. From the Midpoint Theorem, $\triangle DEF \,|||\, \triangle ABC$ and is half the size. It is also oriented 180° relative to $\triangle ABC$. Thus there is a homothetism that maps $\triangle ABC$ to $\triangle DEF$, with scale factor $-\frac{1}{2}$. The centre of similitude must lie on AD, BE and CF, and hence these lines are concurrent.



(b) The homothetism maps AG to DG with scale factor $-\frac{1}{2}$, so $AG : GD = 2 : 1$. The result follows similarly for the other two medians.

(c) The line DO is perpendicular to BC, and hence also to EF. Similarly $EO \perp FD$ and $FO \perp DE$, so O is the orthocentre of $\triangle DEF$. Since the homothetism maps $\triangle ABC$ to $\triangle DEF$, it will also map H to O. This proves the collinearity, and the scale follows as in the previous section.

7.5 Start with an arbitrary pair (Q, R) for which P exists, and construct the excircle C_2 of $\triangle PQR$ opposite P (see diagram).



The incircle and excircle of $\triangle PQR$ must be homothetic, and P is the centre of the homothetism. Now let K be the point of tangency of C with L, and let T be the point diametrically opposite K. The corresponding point to T on C_2 must also be vertically above the centre in the diagram, i.e. it is N. But the line through corresponding points must pass through the centre of the homothetism, so P lies on NT.

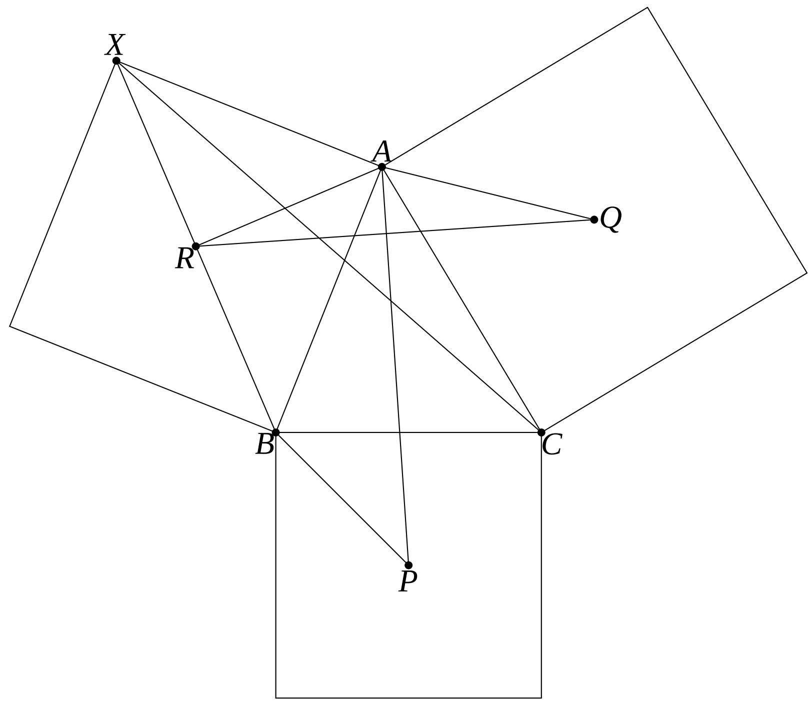
From the solution to problem 6.1 (page 36), we have $QK = RN$, from which it follows that N and K are symmetrically placed about M. But K and M are fixed, so N must be fixed too.

We have now established that any solution P must lie on NT. It is also clear that P must lie strictly beyond T. Conversely, suppose P' is some point on NT beyond T. Let L' be a line through P' and parallel to L, and consider moving a point P along L', finding Q and R on L such that C is the incircle of $\triangle PQR$. When P moves far to the left, the midpoint of QR will be far to the right, and vice versa. Since the midpoint shifts continuously, there is at least one point where it is M. We have shown above that this P must be the intersection of NT with L', namely P', and hence P' satisfies the desired properties. Therefore the locus is the portion of NT that lies strictly beyond T.

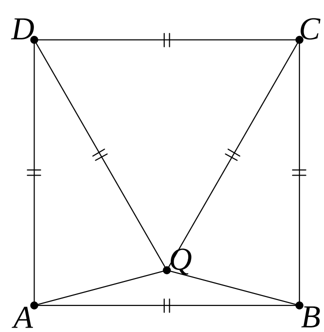
(IMO 1992, question 4)

7.6 Consider the spiral similarity with centre A, rotating clockwise (in the diagram) by 45° and scaling by $\sqrt{2}$. It will map Q to C and R to X. Now consider the spiral similarity with centre B that rotates anti-clockwise by 45° and scales by

$\sqrt{2}$. It will map A to X and P to C. These two similarities thus map AP and QR to the same line. They both scale by the same amount ($\sqrt{2}$) and the difference of their angles is 90°, so AP and QR must be equal and perpendicular.

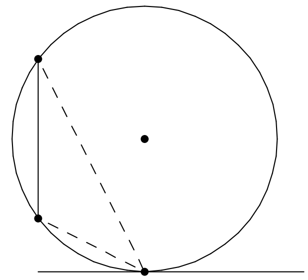


8.1 Construct Q inside the square with $\triangle CDQ$ equilateral. We aim to show that $P = Q$.



Now $\angle QDC = 60^\circ$, so $\angle QDA = 30^\circ$. But $QD = AD$, so $\triangle AQD$ is isosceles and thus $\angle DAQ = 75^\circ$. This makes $\angle BAQ = 15^\circ$, and similarly $\angle ABQ = 15^\circ$. But then triangles ABP and ABQ have two common angles and a common side, so they are congruent. Both P and Q lie on the same side of AB (the inside of the square), so P and Q must be the same. Triangle CDQ is equilateral by construction, so $\triangle CDP$ is equilateral.

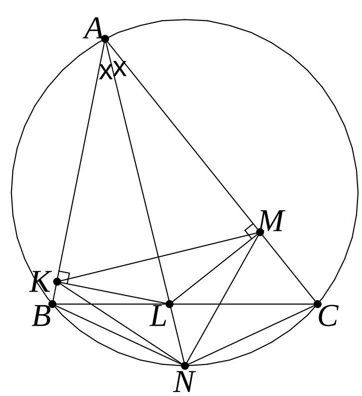
8.2 Construct a circle of radius 5m, with centre 5m above your head height and 4m from the statue. This circle will pass through the head and foot of the statue.



If your head lies on the circle you will have some constant viewing angle θ; with your head inside the circle the angle is larger, and with your head outside the circle it is smaller. But the circle is tangent to the line representing head-height, so the best angle is when your head is at this point of tangency. So you should stand 4m from the statue.

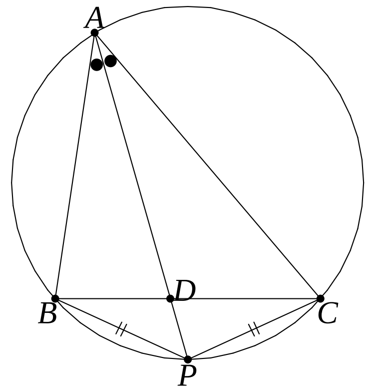
8.3 Firstly note that $\triangle ALK \equiv \triangle ALM$. Hence $AKLM$ is a kite and so $KM \perp AL$; thus $|AKNM| = \frac{1}{2}KM \cdot AN$. Since $ABNC$ is cyclic, $\triangle ABL \,|||\, \triangle ANC$ and hence $AN \cdot AL = AB \cdot AC$. Also, AL is the diameter of the circumcircle of $\triangle AKM$, so $\frac{KM}{AL} = \sin\alpha$. Substituting these into the above gives

$$\begin{aligned}|AKNM| &= \frac{1}{2} \cdot \frac{KM \cdot AB \cdot AC}{AL} \\ &= \frac{1}{2} \cdot AB \cdot AC \cdot \sin\alpha \\ &= |\triangle ABC|\end{aligned}$$



(IMO 1987 Question 2)

8.4 Let D be the point where the angle bisector from A cuts BC.

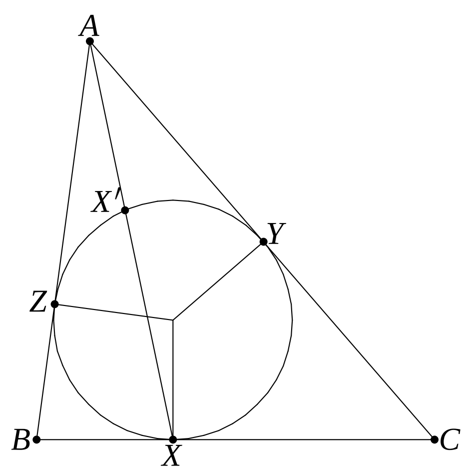


Since $\angle BAD = \angle PAC$ and $\angle DBA = \angle CPA$ we have $\triangle BAD \,|||\, \triangle PAC$. Thus $\frac{c}{BD} = \frac{AP}{PC}$. From exercise 6.4 we have $BD = \frac{ac}{b+c}$. It follows that $AP = \frac{b+c}{a} \cdot PC$. But $PB = PC$ and so from the triangle inequality, $2PC > BC \iff PC > \frac{a}{2}$. Therefore $AP > \frac{b+c}{2}$.

Similarly $BQ > \frac{c+a}{2}$ and $CR > \frac{a+b}{2}$. Adding these inequalities gives the desired result.

(Australian Mathematics Olympiad 1985)

8.5 Firstly note that $AX \cdot AX'$ is the power of A with respect to the incircle, so it is equal to $AZ^2 = x^2$. Thus $a \cdot AX \cdot XX' = a \cdot AX^2 - ax^2$.



We can calculate $a \cdot AX^2$ using Stewart's Theorem:

$$\begin{aligned}
BC(AX^2 + BX \cdot XC) &= AC^2 \cdot BX + AB^2 \cdot CX \\
a(AX^2 + yz) &= b^2 y + c^2 z \\
a \cdot AX^2 &= (x+z)^2 y + (x+y)^2 z - (y+z)yz \\
&= x^2 y + 2xyz + z^2 y + x^2 z + 2xyz + y^2 z - y^2 z - z^2 y
\end{aligned}$$

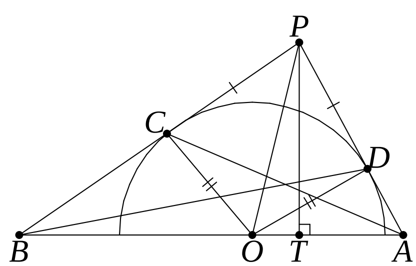
$$
\begin{aligned}
&= x^2(y+z)+4xyz \\
&= ax^2+4xyz.
\end{aligned}
$$

Now we can calculate $a \cdot AX^2 - ax^2$

$$
\begin{aligned}
a \cdot AX^2 - ax^2 &= 4xyz \\
&= \frac{4}{s} \cdot sxyz \\
&= \frac{4}{s} \cdot K^2 \\
&= \frac{4}{s} \cdot rsK \\
&= 4rK \text{ as desired.}
\end{aligned}
$$

(Arbelos May 1987)

8.6 This is a good example of a problem that becomes much easier with a good diagram (the diagram below is intentionally skewed). If AD and BC are extended to meet at P, then it appears that P, E and F are collinear. This would be a useful thing to know, so we attempt to prove it.



Let T be the foot of the perpendicular from P to AB and let O be the centre of the semicircle. $\triangle OCB ||| \triangle PTB$, so $\frac{CB}{TB} = \frac{BO}{BP}$. Similarly $\frac{DA}{TA} = \frac{AO}{AP}$. We want to prove that PT, AC and BD are concurrent, which by the converse of Ceva's Theorem would be true if

$$\frac{PC}{CB} \cdot \frac{BT}{TA} \cdot \frac{AD}{DP} = 1$$

Firstly, $PC = PD$ (equal tangents to the semicircle), and we can substitute the ratios found above to change this to $\frac{BP}{BO} \cdot \frac{AO}{AP} = 1$. However, this is true by the angle bisector theorem (PD is an angle bisector because $\triangle PCO \equiv \triangle PDO$). It follows that E lies on the altitude from A, and $F = T$.

Now notice that PO subtends right angles at C, D and F, so $PCFD$ is a cyclic quad. Thus $\angle DFP = \angle DCP$ and $\angle CFP = \angle CDP$, and since $PC = PD$ it follows that $\angle DFP = \angle CFP$. Therefore EF bisects $\angle CFD$.

(Proposed at IMO 1994)

8.7 The key to this problem is noticing that you can treat triangles ABD and ACE as completely separate, and ignore $\triangle ABC$. The only things these two triangles have in common is the angle at A and the height from A. Let these quantities be θ and h respectively. If we can express $\frac{1}{MB} + \frac{1}{MD}$ in terms of θ and h then we are done.

Let us rename D to C so that we are working with $\triangle ABC$ and can use the usual notation.

$$\begin{aligned}
\frac{1}{MB} + \frac{1}{MC} &= \frac{1}{y} + \frac{1}{z} \\
&= \frac{y+z}{yz} \\
&= \frac{a}{yz} \\
&= \frac{ahrsx}{hrsxyz} \\
&= \frac{ahrsx}{hrK^2} \quad \text{(Heron's Formula)} \\
&= \frac{2K^2x}{hrK^2} \\
&= \frac{x}{r} \cdot \frac{2}{h} \\
&= \frac{2}{h} \cot \frac{\theta}{2}.
\end{aligned}$$

(Proposed at IMO 1993)

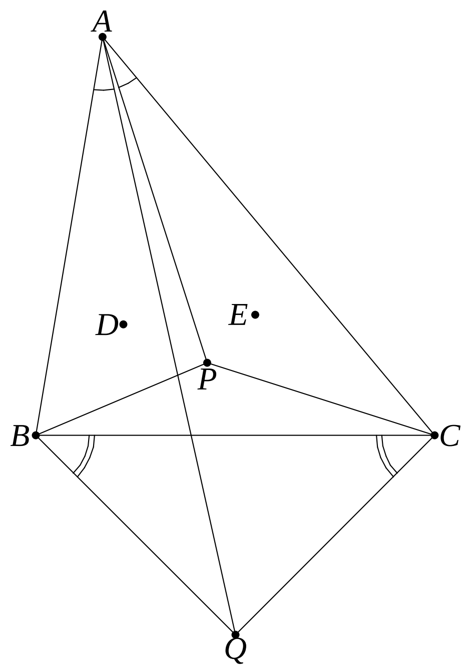
8.8 Construct Q so that $\angle BAQ = \angle PAC$ and $\angle ABQ = \angle APC$. Then by construction, $\triangle ABQ \,|||\, \triangle APC$. Now in $\triangle APB$ and $\triangle ACQ$:

- $\angle BAP = \angle BAC - \angle PAC = \angle QAC$
- $\frac{AC}{AQ} = \frac{AC}{AC \cdot AB/AP} = \frac{AP}{AB}$.

Hence $\triangle APB \,|||\, \triangle ACQ$. Now $\angle CBQ = \angle APC - \angle ABC = \angle APB - \angle ACB =$

$\angle BCQ$, so $\triangle BCQ$ is isosceles. It follows that

$$\frac{AC}{PC} = \frac{AQ}{BQ} = \frac{AQ}{CQ} = \frac{AB}{BP}.$$



Now from the angle bisector theorem, BD will cut AP in the ratio $AB : BP$, and CE will cut AP in the ratio $AC : CP$. Since these ratios are the same, the three lines will be concurrent.

(IMO 1996 Question 2)

10 Recommended further reading

Geometric inequalities often require techniques from the world of standard inequalities. *Inequalities for the Olympiad Enthusiast*, by Graeme West (part of the same series as this booklet) provides some good material in this field.

This booklet is well under 100 pages, and as such cannot do proper justice to the rich field of classical geometry. A highly regarded and very readable reference is *Geometry Revisited*, by Coxeter and Greitzer.

A good source of problems are the yearbooks of the South African training program for the IMO (*South Africa and the* n*th IMO*, for $n \geq 35$). These contain problems and solutions for all the problems used in the training problem, including many good geometry problems.

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