2020 Wits Mathematics Competition
Final Round
Undergraduate
Name:
Institution:

Instructions

This paper is 90 minutes long and consists of ten single answer questions (to be answered in the below table) and two proofs (to be answered on the pages they're written on). In needed, additional sheets of blank paper may be used to finish your solutions. Geometric equipment and language dictionaries are allowed but calculators and other computing devices are not.

"The really unusual day would be one where nothing unusual happens.". — Persi Diaconis

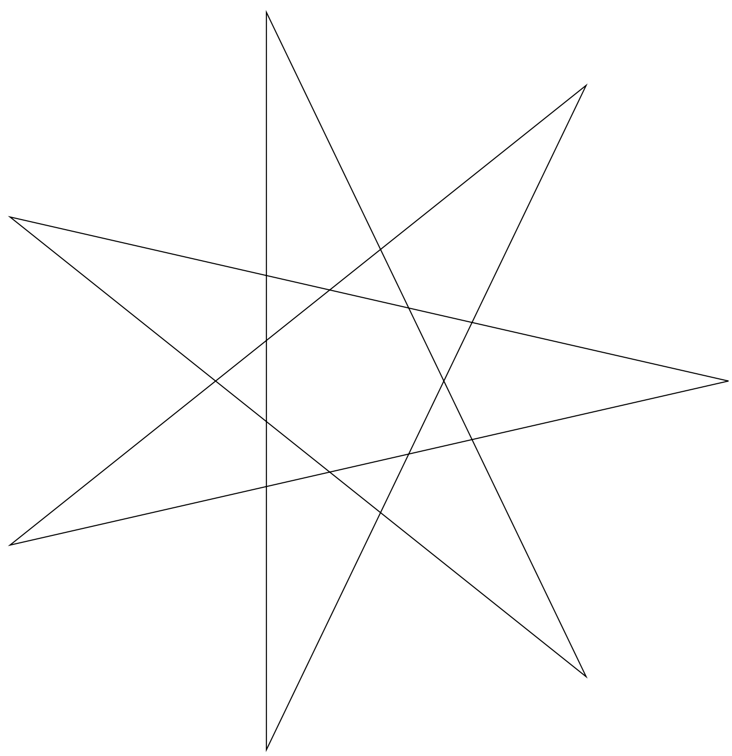
Question	Answer
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	

A. Single Answer Questions

2 Marks

1. The acute triangle ABC has $\hat{C} = 45°$. E is a point on BC where $AE \perp BC$ and D is a point on AC where $BD \perp AC$. Determine the length of DE if $CE = 4$cm and $EB = 3$cm.

2. A seven pointed star is called a heptagram. One is shown below. Find the sum of the angles at the star's tips.



3. How many prime numbers, p, are there such that $p^3 + p^2 + 11p + 2$ is prime?

3 Marks

4. How many positive integers are divisible by 101 and can be written in the form $10^i - 10^j$ for $0 \leq j < i \leq 200$?

5. We have a deck of $n > 1$ cards and a collection of symbols such that the following conditions are satisfied:

   1. Three distinct symbols are drawn on each card.
   2. Any two cards have exactly one symbol in common.
   3. Any two symbols appear together on exactly one card.

What is the value of n?

6. Evaluate the intergral:

$$I = \int \frac{x^3}{a^2 + x^2} dx$$

7. Given that

$$\begin{pmatrix} -2 & -1 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}^{300} = \begin{pmatrix} -1 & -3 & 1 \\ x & 1 & 0 \\ -2 & -3 & 2 \end{pmatrix}$$

find x.

4 Marks

8. Compute the sum:

$$\frac{1}{3\sqrt{2} + 2\sqrt{3}} + \frac{1}{4\sqrt{3} + 3\sqrt{4}} + ... + \frac{1}{100\sqrt{99} + 99\sqrt{100}}$$

9. Evaluate the following integral:

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x + y + z - w}{x + y + z + w} dxdydzdw$$

10. Let R be the region in the first quadrant bounded by the x-axis, the line $2y = x$, and the ellipse $\frac{x^2}{9} + y^2 = 1$. Let R' be the region in the first quadrant bounded by the y-axis, the line $y = mx$, and the same ellipse. Find m such that R and R' have the same area.

B. Proof Questions

11. A certain institution offers a total of six courses. This semester it has twenty students, each of whom takes between zero and six courses (taking no courses is allowed as is taking all six). Is it true that there must exist two courses such that either 5 students take both or 5 students take neither?

12. Let $f : [0, 1) \to [0, 1)$ be the function given by

$$f(x) = \{2x\},$$

where $\{2x\}$ denotes the fractional part of the number $2x$ (e.g. $\{5.67\} = 0.67$ and $\{12\} = 0$). Find the sum of all possible values of $x \in [0, 1)$ such that

$$\underbrace{f \circ f \circ \cdots \circ f}_{10 \text{ times}}(x) = x.$$