WMC 2018 Undergraduate Solutions

Section A

1. **C** There are $2016 \div 4 = 504$ groups of $1, 2, 3, 4$ and an extra 1 and 2. Thus the total sum is is $504(1 + 2 + 3 + 4) + 1 + = 50423$.

2. **E** Since $y < 20$ we have $x \leq 2$ and since x and y are distinct, we have $x = 2$ and $y = 8$ and so $y^2 = 64$.

3. **D** The sum of the numbers from 1 to 11 is $\frac{11 \times 12}{2} = 66$ and the sum of the remaining ten numbers must be 61 so the number that must be removed is $66 - 61 = 5$

4. **D** Let x be the age of the youngest child. Then $x + (x + 1) + (x + 2) = 42 \implies x = 13$. The age of the oldest child is $13 + 6 = 19$.

5. **C** $\frac{40}{100} \times P = \frac{10}{100} \times Q \implies 4P = Q$ and so P is 25% of Q.

6. **A** Let AB be the two digit number in base 7 and so BA is the two digit number in base 5 where $0 \leq A, B \leq 4$. So we have $7 \times A + B = 5 \times B + A \implies 3A = 2B$ and so $A = 2$ and $B = 3$ is the only possible solution.

7. **A** Consider the number formed by the last two digits of 2018 which is 18. Multiplying by 18 and observing the last two digits each time gives the pattern $18, 24, 32, 76, 68, 24, 32, 76, 68, \ldots$ and so from the second term onwards it repeats in cycles of 4, now the remainder when 2018 id divided by 4 is 2 and so the last two digits of 2018^{2018} are 24.

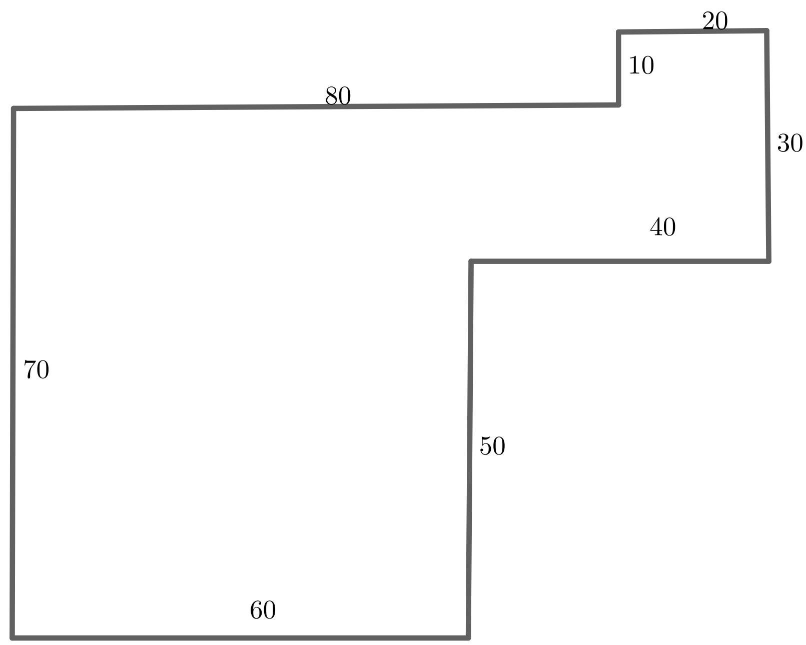
8. **B** Let x and y be the side lengths of the two cubes. Then their surface areas are $6x^2$ and $6y^2$ and so $\frac{6x^2}{6y^2} = (\frac{x}{y})^2 = k \implies \frac{x}{y} = k^{1/2}$. Now the ratio of the two volumes is $\frac{x^3}{y^3} = (\frac{x}{y})^3 = (k^{1/2})^3 = k^{3/2}$.

9. **C** $(2^a)(2^b) = 2^{a+b} = 256 = 2^8 \implies a + b = 8 \implies \frac{a+b}{2} = 4$.

10. **C** Let a be the length of the sides of the triangle. Then the area of the triangle is $\frac{\sqrt{3}}{4}a^2$. Also the semi-perimeter of the triangle is $\frac{3a}{2}$. Now the inradius of the triangle is the area divided by the semi-perimeter and so $9 = \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3a}{2}} = \frac{a\sqrt{3}}{6} \implies a = \frac{54}{\sqrt{3}}$ and so the area is $\frac{54^2\sqrt{3}}{12} = 243\sqrt{3}$.

Section B

11. **5200** The polygon can be uniquely formed as shown in the diagram. The area is then equal to $80 \times 100 - (10 \times 80 + 50 \times 40) = 5200$.



12. **6** Pythagorus on $\triangle ABD$ and $\triangle CBD$ gives $AD = \sqrt{x^2+16}$ and $CD = \sqrt{x^2+81}$. Then Pythagorus on $\triangle ACD$ gives $x^2+16+x^2+81 = 13^2 \implies x = 6$.

13. $\mathbf{\frac{a+2b}{3}}$ We can show by induction that the the nth term is

$$\frac{a(2^{n-1}-2^{n-2}+2^{n-3}-\dots)+b(2^n-2^{n-1}+2^{n-2}-\dots)}{2^n}$$

where the signs alternate. Since we are looking for the limit we do not care of the parity of n. The nth term then simplifies so

$$a(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\dots)+b(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots)$$

which is $\frac{a+2b}{3}$ after you sum the geometric series.

14. $\mathbf{\frac{35}{128}}$ We first choose 4 of the coins to be heads and each coin has a $\frac{1}{2}$ chance of landing on heads or tails. Then the probability is $\binom{7}{4} \times (\frac{1}{2})^7 = \frac{35}{128}$.

15. **196** Consider the sets $\{1,2,3\}$ and $\{6,7,8\}$. We must choose at least element from both of these sets, which we can do in $2^3-1=7$ ways. Now for the set $\{4,5\}$ we can choose any (including the empty set) subset so we have $2^2=4$ choices. So the total number of such subsets is $7 \times 7 \times 4 = 196$.

16. **374** $0,\overline{123} = \frac{123}{1001}(1+\frac{1}{1000}+\frac{1}{1000000}+\dots) = \frac{123}{999} = \frac{41}{333}$ in its lowest form.

17. **50** Suppose without loss of generality that we have a 100 kg orange. Then 80 kg is water, and 20 kg is other content. Now 75% of 80 is 60 so we have $80-60=20$ kg of water left. Then the percentage of water in

the remaining orange is $20 \div (20+20) \times 100 = 50\%$.

18. **222** 2018 in base 3 is 2202202. So consider $k = 1, 2, 3, 4, 5, 6$. Then there are 2^k trinary numbers of length k that satisfy the condition since each of the digits can be 1 or 2. Then suppose the trinary number has seven digits. Then if the first digit is 1 the rest of the numbers can be any of one and two so we get $2^6 = 64$ numbers, and if the first digit is 2, the next has to be 1 or it is too large, and so the rest of the 5 can be filled in $2^5 = 32$ ways. So in total we have $2+4+8+16+32+64+64+32 =$ numbers.

19. $\mathbf{\frac{2^{p-1}}{2^{p-1}-1}}$ Since $p > 1$, the numerator and denominator converge. Let $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots = L$. Now $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots - 2(\frac{1}{2^p} + \frac{1}{4^p} + \dots) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots - \frac{2}{2^p}(\frac{1}{2^p} + \frac{1}{2^p} + \dots)$
$= L - \frac{1}{2^{p-1}}(L) = \frac{L(2^{p-1}-1)}{2^{p-1}}$
Then $\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\dots} = L \times \frac{2^{p-1}}{L(2^{p-1}-1)} = \frac{2^{p-1}}{2^{p-1}-1}$.

20. **-2** Let T_n be the nth term in the in the sequence. Then $T_n = 2^n - 1$ and so $\sum_{k=1}^{n} T_n = \sum_{k=1}^{n} (2^k - 1) = 2^{n+1} - 2 - n \implies R = 1, S = 0, T = -1, U = -2 \therefore R + S + T + U = -2$.

Section C

Question 21

No, it is not possible.
There are 2^{10} different bulb configurations. Now there are also 36 5×5 sub-grids and 64 3×3 sub-grids. Now suppose we choose some of these sub-grids: the number of ways is $2^{64} \times 2^{36} = 2^{100}$. Now label the 1×1 grids by (i, j) where $1 \leq i, j \leq 10$ with the top left square being $(1, 1)$. Choose the 5×5 grids with top left squares $(1,1)$, $(1,4)$, $(4,1)$, $(4,4)$ and the 3×3 grids with top left corners $(1,1)$, $(1,6)$, $(6,1)$, $(6,6)$.This leaves the configuration unchanged. So there exists a configuration of the bulbs that can be reached by two different grid choices, hence there must be one that cannot be reached.

Question 22

The only prime in the sequence is 101. Clearly 101 is prime. We then show that there are no other primes in the sequence.
The nth term in the sequence is equal to $\sum_{k=0}^{n} 10^{2k} = \frac{(10^2)^{n+1}}{99} = \frac{(10^{n+1}-1)(10^{n+1}+1)}{99}$. Now $10^{n+1} - 1$ is divisible by 9 and let $\frac{10^{n+1}-1}{9} = K_n \in \mathbb{N}$. So the nth term in the sequence is $K_n \times \frac{10^{n+1}+1}{11}$ which we know is a positive integer, and since 11 is prime, it must divide either K_n or $10^{n+1} - 1$. Now since $n \geq 2$ both K_n and $10^{n+1} - 1$ are greater than 11 and so if 11 divides one of them, it is in the form $11a$ for some positive integer a greater than 1. Then the nth term is the product of two numbers greater that 1 and cannot be prime.