COMPLEX GEOMETRY FOR THE OLYMPIAD ENTHUSIAST

DIRK LAURIE

INTRODUCTION

There are many ways to solve problems in geometry. Traditional synthetic geometry often produces very short and elegant proofs if only one could find the correct construction to make, or the correct intermediate hypothesis to prove.

Other methods, like trigonometry, vectors, transformation and inversion, occasionally provide short neat proofs. If all inspiration fails, one can always resort to coordinate geometry and try to grind out a solution, but this seldom actually works. You just lose too much when separating the coordinates.

In an actual Olympiad paper, there is not enough time to try out all the various approaches. But there is good news for you: there is a single method that combines the virtues of trigonometry, vectors and coordinates. This all-in-one tool is analytical geometry using complex numbers. If it doesn't work out with complex numbers, it probably would not have worked out with any of those other alternative methods either.

Complex geometry works because: the coordinates of a point are kept together, as in vector geometry; it is easy to describe translation, rotation and reflection, as in transformation geometry; trigonometric identities are in effect being applied all the time but you seldom need to be conscious of them; and all this is achieved by arithmetic operations on points. There are certain problems where it gives particularly neat and short solutions; you will meet several such here. Others, where it is clumsy and cumbersome, also occur, but not here.

You need to acquire virtuosity in handling the complex geometry tools before they will become useful to you. There is no substitute for practice, and you should rework by complex numbers problems that you encounter elsewhere, even those that you could solve by other methods.

The problems in this booklet fall into three categories: exercises in the techniques, supplements to the theory, and actual competition level

problems. To get full value from studying this material, you need to do them all. If you get stuck, there are hints at the back to some of the problems.

1. COMPLEX NUMBERS

You know from Cartesian coordinate geometry that any point in the plane can be represented by a pair of real numbers. Identify the point (x, y) with the complex number $z = x + yi$ where i is an imaginary quantity with the curious property that $i^2 = -1$ (obviously i is not a real number) which otherwise satisfies all the normal manipulation rules of algebra. Therefore addition and multiplication of complex numbers are defined as follows: if $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then

$$z_1 + z_2 = (x_1 + y_1) + (x_2 + y_2)i; \tag{1}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i. \tag{2}$$

Notation: in algebra, when you write two symbols together, as in ab, it means multiplication of a and b, and that rule is followed here too, most of the time. But since this booklet deals with geometry, we make one exception: when all the symbols involve capital letters that are associated with points, then it may mean some geometric object. For instance, A_1A_2, ABC, $PQRS$ etc. usually denote lines, angles, polygons, etc. It will always be clear from the context when this happens.

The formula for adding complex numbers is simply standard vector addition of points, when each point is identified with the vector from the origin to that point. Geometrically, addition is *translation*: when a given point is added to each of a set of points, the set of points just gets moved around with no change in shape, orientation etc.

Multiplication looks complicated, but becomes easier to understand when you use polar coordinates , so that $x = r\cos\theta$ and $y = r\sin\theta$, where θ is taken to be the anti-clockwise angle from the positive x-axis to the ray from the origin through (x, y). The *modulus* of a complex number is defined as $|z| = r$. The polar angle when taken in the range $-180° < \theta \le 180°$ is the *argument* of a complex number and denoted by $\arg z$. Strictly speaking θ should be in radians, but we simply define the $°$ symbol so that $m° = m\pi/180$ and continue talking in terms of *degrees*.

The polar representation gives $z = r(\cos\theta + i\sin\theta) = re^{\theta i}$. Those who have learnt some calculus will know what e means, but that is not relevant here: we think of $e^{\theta i}$ as a convenient shorthand notation for

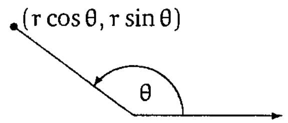

FIGURE 1. Polar coordinates

$\cos\theta + i\sin\theta$, easily remembered because the exponential law

$$e^{\theta_1 i} e^{\theta_2 i} = e^{(\theta_1+\theta_2)i}$$

is satisfied (check it using trigonometry). Therefore

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) = r_1 r_2 e^{(\theta_1+\theta_2)i}; \tag{3}$$

$$z_1/z_2 = r_1/r_2 (\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) = r_1/r_2 e^{(\theta_1-\theta_2)i}. \tag{4}$$

In words: to multiply two complex numbers, you multiply the radii and add the angles. If one of the points (say z_1) is on the *unit circle* $r = 1$, then the effect of multiplication by z_1 is rotation of the other point by the angle θ_1 around the origin. If one of the points (say $z_2 = c$) is real, then the effect of multiplication is scaling by a factor c.

A word of caution about the argument: equations like

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$
$$\arg(z_1/z_2) = \arg z_1 - \arg z_2$$

are only true when the right-hand side lies in the correct range. Otherwise all we can say is that the equation is true modulo 360°.

When two angles differ by a multiple of 360°, they look the same when sketched and are represented by the same complex number, but they may behave differently in computation. For example, one can solve the equation $z^5 = 1$ as follows: let $z = \cos\theta + i\sin\theta$, then $z^5 = \cos 5\theta + i\sin 5\theta = 1 = \cos 0 + i\sin 0$. Therefore $\theta = 0$ gives a solution $z = 1$. But it is also true that $1 = \cos 360° + i\sin 360°$, which leads to $\theta = 72°$, etc.

1.1. Conjugates.

The *conjugate* $\bar{z}$ of a complex number z is defined by

$$\overline{x + iy} = x - iy.$$

Note that $r^2 = |z|^2 = \bar{z}z$. The geometrical interpretation is that $\bar{z}$ is the reflection of z in the x-axis.

Nothing is lost when you work with complex numbers and their conjugates instead of the x and y coordinates separately. You can always if needed recover the *real* and *imaginary* parts of z by the formulas

$$\operatorname{Re} z = x = (z + \bar{z})/2, \tag{5}$$

$$\operatorname{Im} z = y = (z - \bar{z})/(2i) = i(\bar{z} - z)/2, \tag{6}$$

and therefore it is never necessary to use x and y explicitly.

You should convince yourself of the identities

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,$$
$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$$

so that the conjugate of an expression involving the four basic arithmetic operations can be calculated by conjugating everything inside and then doing the arithmetic. An expression such as $\bar{z}z$ which remains unchanged when conjugated, must always be real.

There are several important reasons for introducing conjugates:

- It simplifies division so you need only divide by real numbers, since
$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{r^2}.$$
- Take a polynomial equation with real coefficients, e.g. $z^4+z+1 = 0$. Conjugate everything to obtain $\bar{z}^4+\bar{z}+1 = 0$. You get the same equation as before with $\bar{z}$ as the unknown. Therefore every root either satisfies $z = \bar{z}$ (i.e. z is real) or there is another root $\bar{z}$.
- It is inconvenient to work with equations in which some quantities are complex but others are restricted to be real, such as:
$$z = z_1 + t(z_2 - z_1) \text{ where } t \text{ is real.}$$
You easily lose track of the extra information, and anyhow the variable t is an unnecessary extra symbol. Instead, write the statement "$(z - z_1)/(z_2 - z_1)$ is real" in the form
$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$
in which all the quantities may be as complex as they please.

Problem 1. *The "n-th root of unity" is defined by*

$$w_n = e^{2\pi i/n}.$$

Show that:

(1) *the numbers* $1, w_n, w_n^2, \dots, w_n^{n-1}$ *lie at the vertices of a regular polygon;*
(2) $1 + w_n + w_n^2 + \dots + w_n^{n-1} = 0$;
(3) *when* $n = 2m$ *where* m *is an odd number, then*

$$1 - w_{2m} + w_{2m}^2 - w_{2m}^3 + \dots + w_{2m}^{m-1} = 0.$$

Problem 2. *For all complex numbers* $z = x + iy$ *not on the negative real axis, we define* $\sqrt{z} = u + vi$ *so that* $u > 0$; *the other solution to* $w^2 = z$ *is obviously* $w = -\sqrt{z}$. *Show that* $u = \sqrt{(r+x)/2}$ *and* $v = \pm\sqrt{(r-x)/2}$, *where* v *has the same sign as* y.

Problem 3. *If* $\tan 4\theta = \frac{12}{5}$, *calculate all possible values of* $\tan\theta$.

2. Transformations

We have already seen that translation, rotation, scaling and reflection are readily expressible in terms of complex arithmetic, so you can do transformation geometry by complex numbers. This is a powerful tool in finding quick solutions to some problems.

The basic principle is that the *linear transformation* $z \mapsto az + b$ is a combination of translation, rotation and scaling, which are all operations that preserve lengths and angles. Many problems are much easier when important points lie on the origin or on the coordinate axes. If we do not have the freedom to choose the coordinate system, we can always transform to the one we like, do the problem in an easy way, and then transform the result back to the original system.

Here is a simple but important example.

Perpendicular from point to line. It is easy to find the foot D of the perpendicular from C to the line AB when $A = 0$ and B is real: the answer is $D = \operatorname{Re} C$. The linear transformation that takes A to 0 and B to 1 is $z \mapsto (z - A)/(B - A)$. Therefore in the general case,

$$\frac{D-A}{B-A} = \operatorname{Re}\frac{C-A}{B-A}$$

$$\iff D = A + (B-A)\operatorname{Re}\frac{C-A}{B-A} \tag{7}$$

Some complex expressions are *invariant* under certain transformations. For example, $A - B$ is invariant under translation, because the transformation $z \mapsto z + w$ changes it to $(A + w) - (B + w) = A - B$; $\overline{A}B$ is invariant under rotation, since the transformation $z \mapsto wz$ with $|w| = 1$

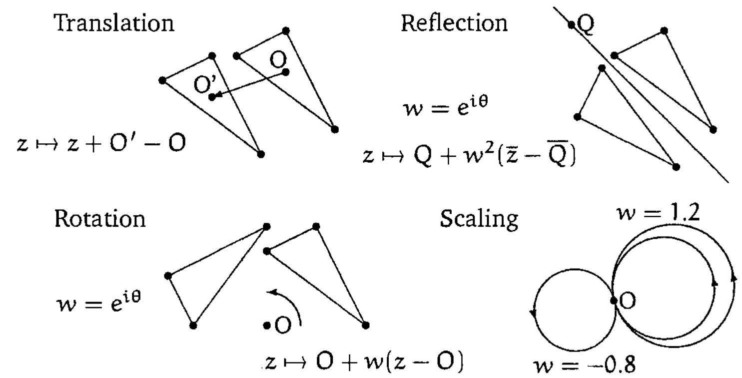

FIGURE 2. Transformations and complex arithmetic. Rotation and scaling have the same formula, but rotation requires $|w| = 1$ where scaling requires w to be real.

changes it to $\overline{wA}wB = \overline{A}\overline{w}wB = \overline{A}B$ because $\overline{w}w = |w|^2 = 1$; and A/B is invariant under rotation and scaling, since the transformation $z \mapsto cz$, where c is any complex number, takes it to itself.

Invariant expressions are important in deriving formulas. For example, the area of a rectangle with vertices at $D(0,0)$, $A(a,0)$, $B(0,b)$ and $C(a,b)$ is ab. There are several complex expressions that all give this result, among them $-iAB$ and $i\overline{B}A$. But $-iAB$ is not invariant under rotation, whereas $i\overline{B}A$ is. Therefore $i\overline{B}A$ gives the area of any rectangle $DACB$ when $D = 0$. Another possible expression is $i(\overline{B} - \overline{D})(A - D)$, which is invariant under translation and rotation, and therefore gives the area of any rectangle $DACB$. Note that the area of $DBCA$ comes out negative: in the next section we discuss what that means.

Problem 4. *Write expressions for the following operations:*

(1) *Rotate* z *clockwise by* $20°$ *around the point* P.
(2) *Reflect the point* A *in the line through* P *and* Q.

Problem 5. *Show that another formula for dropping a perpendicular to a line is*

$$D = C - i(B-A)\,Im\,\frac{C-A}{B-A}.$$

Problem 6. *It is known from transformation geometry that*

(1) *Reflection in the line AB followed by reflection in the line CD is equivalent to a rotation.*
(2) *Rotation by θ_1 around z_1 followed by ~~reflection~~ rotation? by θ_2 around z_2 is equivalent to a single rotation.*

Prove both statements by calculating the centre and angle of rotation in each case.

3. Geometrical measurements

Lengths are always positive, but sides are have *direction*: AB stands for the directed line segment from A to B, which is the vector $B - A$. The length of AB is $|B - A|$.

For any point P, it is a matter of simple arithmetic that

$$\frac{A-P}{P-B} = \frac{m}{n} \iff P = \frac{mA+nB}{m+n}.$$

When P lies on the line segment AB, it therefore divides AB in the ratio $m : n$.

Angles and areas may be negative. A positive angle is measured in an anti-clockwise direction, a negative angle in a clockwise one. The directed angle ABC is the amount by which BC needs to be rotated until it lies on BA. It is usually convenient to think of a reflex angle θ as a non-reflex angle $\theta \pm 360°$ of opposite sign.

An angle is calculated by

$$A\widehat{B}C = \arg{(A-B)/(C-B)},$$

not by $\arg(A-B) - \arg(C-B)$, in order to make sure that we get an angle in the correct range. Generally

$$C\widehat{B}A = -A\widehat{B}C$$

except when both angles are 180° : the definition of 'argument' does not allow $-180°$.

The sign of the area of a figure depends on how its perimeter is specified. We start with triangles. If the angle ABC is such that $A\widehat{B}C > 0$, then also its area $[ABC] = [BCA] = [CAB] > 0$; but $[ACB] = [CBA] = [BAC] = -[ABC] < 0$. (The sine area formula $[ABC] = \frac{1}{2} bc \sin A\widehat{B}C$ thus remains true even when $A\widehat{B}C < 0$. But you won't need this formula: there are easier ways to calculate areas.)

There are two other ways to tell the sign of an area. The notation ABC indicates that the directed sides are AB, BC and CA. If you *traverse*

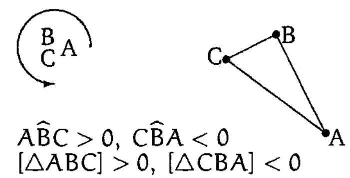

FIGURE 3. Positive angles and areas always work with points taken anti-clockwise.

the perimeter of a triangle (i.e.. walk around it along the directed sides until you get back to where you started), the enclosed area is positive if you have gone through an angle of 360° and negative if you have gone through $-360°$. If you are too lazy to walk, just stand on the perimeter looking forwards: the area of the triangle is positive if its interior is to your left.

These two ways of telling the sign of an area apply to any non-intersecting figure, even curved ones. They also apply to a figure with holes in, as long as the perimeter of each hole is specified so that the interior of the figure lies the same way (always or never to the left) on each directed line segment.

The area of a triangle is calculated by the formula

$$[ABC] = \tfrac{1}{2}\,\mathrm{Im}(\overline{A}C + \overline{B}A + \overline{C}B). \tag{8}$$

You will be asked to prove this formula in the exercises. Alternative forms that are occasionally more convenient are:

$$[ABC] = \frac{i}{4}\left(\overline{C}(B-A) + \overline{B}(A-C) + \overline{A}(C-B)\right) \tag{9}$$

$$= \tfrac{1}{2}\,\mathrm{Im}\left((\overline{B}-\overline{A})(C-A)\right). \tag{10}$$

If $C\widehat{A}B = 90°$, $(\overline{B}-\overline{A})(C-A)$ is purely imaginary and we have

$$[ABC] = -\frac{i}{2}(\overline{B}-\overline{A})(C-A).$$

The area formula generalizes to an arbitrary non-intersecting polygon $A_1A_2\ldots A_n$:

$$[A_1A_2\ldots A_n] = \frac{1}{2}\sum_{k=1}^{n} \mathrm{Im}(\overline{A}_k A_{k-1}), \tag{11}$$

where $A_0 = A_n$. If the polygon is convex and the origin is inside, you can immediately recognize the formula as the sum of the areas of the triangles $A_{k-1}A_kO$. If not, our convention about negative areas ensures that the formula is still correct.

A useful application of signed areas is *area coordinates*. Let P be any point, and define

$$\alpha = [PBC]/[ABC],\ \beta = [PCA]/[BCA],;\gamma = [PAB]/[CAB].$$

Then it can be shown that

$$P = \alpha A + \beta B + \gamma C,\ \text{where } \alpha,\beta,\gamma \in \mathbb{R} \text{ and } \alpha+\beta+\gamma = 1. \tag{12}$$

Problem 7. *Prove the formula for the area of a triangle.*

Problem 8. *Verify (12) using reversible steps in your argument, thereby showing that if P is given by (12), then α, β and γ are uniquely determined.*

4. Geometric relations involving angles

To test in general whether two directed angles are equal, the requirement is that

$$A\hat{B}C = D\hat{E}F \iff \frac{A-B}{C-B}\Big/\frac{D-E}{F-E} \text{ is real and positive.}$$

The 'and positive' part is annoying since it becomes difficult to manipulate the relation. It may be dropped when we do not need to distinguish between angles θ and $\theta + 180°$.

Two lines AB and CD are *parallel* when $\arg(A-B) = \arg(C-D)$ or $\arg(A-B) = \arg(C-D) \pm 180°$; combine the cases by observing that they both say $(A-B)/(C-D)$ is real. Therefore

$$AB \parallel CD \iff (A-B)/(C-D) = (\overline{A}-\overline{B})/(\overline{C}-\overline{D}). \tag{13}$$

Three points A, B and C are *collinear* when $AB \parallel BC$.

Two lines AB and CD are *perpendicular* when $\arg(A-B) = \arg(C-D) \pm 90°$; combine the cases by observing that they both say $(A-B)/(C-D)$ is purely imaginary. Therefore

$$AB \perp CD \iff (A-B)/(C-D) = -(\overline{A}-\overline{B})/(\overline{C}-\overline{D}). \tag{14}$$

Two geometric figures are *homothetic* with centre A if one can be made to coincide with the other by scaling with fixed point A. In other words, there exists a real constant c such that for any point z on the first figure, there is a point z' on the second figure so that

$$z' - A = c(z - A).$$

In complex geometry, it is convenient to define triangles ABC and PQR as *similar* (written $\triangle ABC \sim \triangle PQR$) only when they can be made homothetic by rotating one of them. This requires that they are traversed in the same direction, and one can be made to coincide with the other by some combination of rotation, translation and scaling. There must be certain complex numbers $w \neq 0$ and z such that

$$A = z + wP, \quad B = z + wQ, \quad C = z + wR.$$

Eliminating w and z from these three equations, we find that

$$\triangle ABC \sim \triangle PQR \iff \frac{A-B}{C-B} = \frac{P-Q}{R-Q}.$$

If they are traversed in opposite ways, we need to reflect one of them to make them the same. So the test becomes

$$\triangle ABC \sim_{\text{rev}} \triangle PQR \iff \frac{A-B}{C-B} = \frac{\overline{P}-\overline{Q}}{\overline{R}-\overline{Q}}.$$

In Euclidean geometry we usually define similarity of triangles by

$$\triangle ABC \mathrel{|||} \triangle PQR \iff \triangle ABC \sim \triangle PQR \text{ or } \triangle ABC \sim_{\text{rev}} \triangle PQR.$$

This concept is seldom required in complex geometry.

Triangles are congruent if one can be made to coincide with the other by some combination of rotation, translation and reflection: like being similar, but with scale factor 1. This gives

$$\triangle ABC \equiv \triangle PQR \iff \frac{C-A}{R-P} = \frac{B-A}{Q-P} = w \text{ with } |w| = 1.$$

As in the case of similar triangles, for full compatibility with the traditional definition of congruence one needs to define a $\equiv_{\text{rev}}$ relation. However, congruent triangles are not important in complex geometry, since the things we use them for (proving equality of lengths and angles) can be done directly once we have the points.

Problem 9. *Three equilateral triangles ABP, CDP and EFP have a common vertex P such that $B\hat{P}C + D\hat{P}E + F\hat{P}A = 180°$. L, M and N are the midpoints of BC, DE and FA respectively. Prove that LMN is an equilateral triangle. [POTW]*

Problem 10. *Prove Napoleon's theorem, which says that if you draw equilateral triangles on all three sides of an arbitrary triangle, their centroids form an equilateral triangle.*

Problem 11. *Three circles, with radii* p, q, r, *and centres* A, B, C *respectively, touch one another externally at points* D, E, F. *Prove that the ratio of the areas of* $\triangle DEF$ *and* $\triangle ABC$ *equals*

$$\frac{2pqr}{(p+q)(q+r)(r+p)}.$$

[SAMO 1995]

5. Equations of lines and circles

In complex geometry, all equations are written in terms of z and $\bar{z}$, not in terms of x and y. This may not seem to be a great advantage, but there is an inherent symmetry between a complex number and its conjugate that does not exist between the coordinates themselves. Most of the equations are *self-conjugate*: when you conjugate everything, you just get the original equation again. When you have two equations, you solve only for z.

5.1. **Lines.** A general point z on the line AB is *collinear* with A and B, so from (13) we get

$$(z-A)/(A-B) = (\bar{z}-\bar{A})/(\bar{A}-\bar{B}).$$

Putting $s = A - B$, we obtain

$$\frac{z}{s} - \frac{\bar{z}}{\bar{s}} = \frac{A}{s} - \frac{\bar{A}}{\bar{s}}. \tag{15}$$

In general, we denote a line through A parallel to s by $\ell(A \parallel s)$.

It is sometimes more convenient to descibe a line in terms of a vector perpendicular to it. Put $w =$ is in (15) to obtain

$$\frac{z}{w} + \frac{\bar{z}}{\bar{w}} = \frac{A}{w} + \frac{\bar{A}}{\bar{w}}.$$

In general, we denote a line through A perpendicular to w by $\ell(A \perp w)$.

When $\ell(A \perp w)$ does not pass through zero, it contains a point p closest to zero, so that p is perpendicular to $\ell(A \perp w)$. Choosing $A = w = p$, we get

$$\frac{z}{p} + \frac{\bar{z}}{\bar{p}} = 2. \tag{16}$$

This equation fully characterizes a line in terms of only one point.

When we arrive at an equation of the form $az + b\bar{z} = c$ by other means, it is useful to know whether the equation does in fact describe

a line. Conjugate everything to obtain $\bar{a}\bar{z} + \bar{b}z = \bar{c}$. This equation only describes a line if it says exactly the same as the original one. Therefore we need $\bar{b}/a = \bar{a}/b$, i.e. $|a| = |b|$, and $\bar{a}\bar{c} = bc$. If the first of these conditions is not satisfied, we can solve the two equations for z; if the first is satisfied but not the second, the two equations are in conflict.

5.2. **Circles.** The equation of a circle with radius r and centre Q is very easy

$$|z-Q|^2 = r^2.$$

We will denote this circle by $\kappa(Q, r)$. Multiplying out, we obtain

$$z\bar{z} - \bar{Q}z - \bar{z}Q = r^2 - Q\bar{Q}. \tag{17}$$

From this you can see that an equation of the form $z\bar{z} + az + b\bar{z} + c = 0$ describes a circle if and only if $b = \bar{a}$, c is real, and $|b|^2 - c > 0$.

To find the *circumcircle* of $\triangle ABC$, we substitute the three vertices into (17) and subtract one equation from both others to obtain

$$\bar{Q}(A-B) + Q(\bar{A}-\bar{B}) = A\bar{A} - B\bar{B};$$
$$\bar{Q}(A-C) + Q(\bar{A}-\bar{C}) = A\bar{A} - C\bar{C}.$$

Eliminating $\bar{Q}$ and regroup:

$$Q = \frac{A(B-C)\bar{A} + B(C-A)\bar{B} + C(A-B)\bar{C}}{(B-C)\bar{A} + (C-A)\bar{B} + (A-B)\bar{C}}.$$

The radius is found from e.g. $r^2 = (A-Q)(\bar{A}-\bar{Q})$, which simplifies to

$$r = \left|\frac{(A-B)(B-C)(C-A)}{(B-C)\bar{A} + (C-A)\bar{B} + (A-B)\bar{C}}\right|$$

To find the *incircle*, the easiest method is by area coordinates. When P is the incentre I, the triangles PBC, PCA and PAB have equal height and therefore the area is proportional to the bases. We obtain

$$I = \frac{A|B-C| + B|C-A| + C|A-B|}{|B-C| + |C-A| + |A-B|}.$$

The *excircles* are found similarly. E.g. for the excircle opposite to A, the triangle PBC is traversed in the opposite direction and therefore has negative area proportional to $-|B-C|$. We obtain

$$I_A = \frac{-A|B-C| + B|C-A| + C|A-B|}{-|B-C| + |C-A| + |A-B|} \text{ etc.}$$

Problem 12. *Find the radius of the incircle by calculating* $|I - R|$. *Manipulate the result to obtain an expression that is symmetric in* A, B *and* C.

5.3. **Tangents.** Suppose that A is on the circle $\kappa(Q,\tau)$. If $Q = 0$, finding the tangent is trivial, since A is the least point on the tangent. This would give $z/A + \bar{z}/\bar{A} = 2$. To move the origin to 0, do a translation:

$$\frac{z-Q}{A-Q} + \frac{\bar{z}-\bar{Q}}{\bar{A}-\bar{Q}} = 2. \tag{18}$$

Suppose that A is on the circle $\kappa(Q,\tau)$, and we wish to find the centre of the circle $\kappa(Q',\tau')$ of radius τ' that touches $\kappa(Q,\tau)$ at A. The easiest method is by homothety: if z' is a point on $\kappa(z',\tau')$, then there is a point z on $\kappa(Q,\tau)$ such that $(z' - A)/\tau' = \pm(z - A)/\tau$ (the plus sign applies when the circles touch internally and the minus sign when they touch externally). The homothety applies to the whole figure, including the centre. Therefore $Q' = A \pm (\tau'/\tau)(Q - A)$.

6. Constructions

Constructions involve setting up and solving equations. There is not much point in trying to remember formulas even in simple, common cases such as the intersection of two lines. You must be able to apply the basic principles to the case at hand.

6.1. **Intersection of diagonals of quadrilateral.** The problem is of course the same as finding the intersection of two lines, but it is easier to think in terms of a quadrilateral $ABCD$. Let X be the intersection of AC and BD. In order to retain the symmetry of the four given points, set up the collinearity equations in the form

$$\frac{X-A}{X-C} = \frac{\bar{X}-\bar{A}}{\bar{X}-\bar{C}}$$
$$\iff X(\bar{A}-\bar{C}) - \bar{X}(A-C) = \bar{A}C - A\bar{C},$$

and similarly

$$X(\bar{C}-\bar{D}) - \bar{X}(C-D) = \bar{C}D - C\bar{D}.$$

Solving for X, we find

$$X = \frac{(\bar{A}B - A\bar{B})(C-D) - (\bar{C}D - C\bar{D})(A-B)}{(A-B)(\bar{C}-\bar{D}) - (C-D)(\bar{A}-\bar{B})}.$$

The answer can be simplified quite a bit by noting that there are many subexpressions of the form $z - \bar{z} = 2i \operatorname{Im} z$. We obtain

$$X = \frac{(B-D)\operatorname{Im}(\bar{C}A) - (A-C)\operatorname{Im}(\bar{D}B)}{\operatorname{Im}\left((\bar{B}-\bar{D})(A-C)\right)}$$
$$= \frac{A\operatorname{Im}(\bar{B}D) + B\operatorname{Im}(\bar{C}A) + C\operatorname{Im}(\bar{D}B) + D\operatorname{Im}(\bar{A}C)}{\operatorname{Im}(\bar{A}D + \bar{B}A + \bar{C}B + \bar{D}C)}$$
$$= \sum_{k=1}^{4} A_k \operatorname{Im}(\bar{A}_{k+1}A_{k-1}) \Big/ \sum_{k=1}^{4} \operatorname{Im}(\bar{A}_k A_{k-1}), \tag{19}$$

where $A_0 = A_4$ and $A_1 = A_5$. The last form of the formula is almost simple enough to remember after all: the denominator is just the area of the quadrilateral.

In an actual calculation, one would hope that the four given points are not totally independent, and that further simplifications occur.

6.2. **Intersection of line and circle.** This one is tricky. You can eliminate $\bar{z}$ from the two equations to get a quadratic in z, but this quadratic always has a solution (since we are working with complex numbers) while the line may be missing the circle. To be safe, first calculate the point P on the line that is closest to the centre of the circle $\kappa(Q,\tau)$. If $|P - Q| > \tau$, there is no intersection.

6.3. **Intersection of two circles.** First find the *radical axis*, which is the line through the intersection points of $\kappa(Q_1,\tau_1)$ and $\kappa(Q_2,\tau_2)$. Subtract the equation for the second circle from that of the first. The quadratic terms cancel, and we obtain

$$(Q_2 - Q_1)\bar{z} + (\bar{Q}_2 - \bar{Q}_1)z = \tau_1^2 - \tau_2^2 - |Q_1|^2 + |Q_2|^2 \tag{20}$$

which is the equation of a line. Now find the intersection of either circle with the radical axis.

The radical axis is the common tangent when the circles touch. It is still defined by (20) even when the circles are disjunct, but its geometrical meaning is then more complicated.

7. Advanced geometric relations

7.1. **Concurrent lines.** To test in general whether lines are concurrent, you calculate the intersection of two of the lines (see Section 6) and then test whether that point is collinear with the two remaining points, which can be quite tedious. There is however a neat shortcut when the lines are in closest-point form.

Put $a = 1/p$ in (16) to obtain

$$a z + \bar{a}\bar{z} = 2. \tag{21}$$

When we speak of "the line a," we mean the line given by equation (21).

Equation (21) remains unchanged if we exchange a (which stands for a line) and z (which stands for a point). Therefore three equations saying that three lines a, b and c all pass through a point z look exactly like three equations saying that three points a, b, and c all lie on a line z. So the three lines a, b and c are concurrent if and only if the three points a, b and c are collinear. But we already have a test for collinearity by using equation (13). This gives

$$\text{Lines } a, b \text{ and } c \text{ are concurrent} \iff \frac{a-b}{\bar{a}-\bar{b}} = \frac{c-b}{\bar{c}-\bar{b}}. \tag{22}$$

When the lines AD, BE and CF join each vertex of $\triangle ABC$ to a point on the opposite side, you can use the complex form of *Ceva's theorem*, which states:

When D, E and F are on BC, CA and AB respectively, then

$$\text{AD, BE and CF are concurrent} \iff \frac{D-B}{D-C}\frac{E-C}{E-A}\frac{F-A}{F-B} = -1. \tag{23}$$

Menelaus' theorem looks very similar: When D, E and F are on BC, CA and AB respectively, then

$$\text{D, E and F are collinear} \iff \frac{D-B}{D-C}\frac{E-C}{E-A}\frac{F-A}{F-B} = 1. \tag{24}$$

Problem 13. *Given* $\delta = \frac{D-B}{D-C}$, $\eta = \frac{E-C}{E-A}$ *and* $\phi = \frac{F-A}{F-B}$ *such that* $\delta\eta\phi = -1$, *calculate* α, β *and* γ *such that the intersection* P *of* AD, BE *and* CF *is given by*

$$P = \alpha A + \beta B + \gamma C.$$

7.2. **Cocylic points.** Four points A, B, C and D are cocylic if and only if either $A\hat{B}C = A\hat{D}C$, or $A\hat{B}C$ and $C\hat{D}A$ are supplementary. Equal angles arise when

$$\arg(A-B)/(C-B) - \arg(A-D)/(C-D) = 0$$
$$\iff \arg\left(\frac{(A-B)(C-D)}{(C-B)(A-D)}\right) = 0.$$

Supplementary angles arise when

$$\arg(A-B)/(C-B) + \arg(C-D)/(A-D) = 180°$$
$$\iff \arg\left(\frac{(A-B)(C-D)}{(C-B)(A-D)}\right) = 180°.$$

The two cases can be combined by saying that

$$\text{A, B, C and D are cocylic} \iff \frac{A-B}{A-D} \Big/ \frac{C-B}{C-D} \text{ is real.} \tag{25}$$

The expression in (25) is called the *cross-ratio* of the four points and denoted by $(A, C; B, D)$.

Note that the degenerate case where four points are collinear also counts as cocyclic. (Think, if you will, that the circle has infinite radius.)

Problem 14. ABC *is a triangle with sides 1, 2, and* $\sqrt{3}$. *Determine the smallest possible area of an equilateral triangle with one vertex on each of the sides of* ABC.
[SAMO 1996]

Problem 15. *The convex quadrilateral* ABCD *has* AC $\perp$ BD *and the perpendicular bisectors of* AB *and* CD *meet at a point* P *inside* ABCD. *Show that* ABCD *is cyclic if and only if* [ABP] = [CDP].
[IMO 1998]

Problem 16. *Circles* k_1 *and* k_2 *are drawn so that* k_2 *passes through the centre of* k_1. *The circles intersect in points* A *and* B. *A third circle* k_3 *touches* k_1 *in* C *and* k_2 *in* D *so that* k_1 *and* k_2 *are inside* k_3. AB *is extended to meet* k_3 *in* E *and* F. *The lines* DE *and* DF *intersect* k_2 *respectively in* G *and* H. *Prove that* GH *is tangent to* k_1.
[IMO 1999]

Problem 17. A, B, C D, E *and* F *lie in that order on the circumference of a circle. The chords* AD, BE *and* CF *are concurrent.* P, Q *and* R *are the midpoints of* AD, BE *and* CF *respectively. Two further chords* AG $\parallel$ BE *and* AH $\parallel$ CF *are drawn. Prove that* $\triangle PQR \ ||| \ \triangle DGH$.
[SAMO 1998]

HINTS

Problem 5: Which linear transformation maps C to 0 and $B - A$ to 1?

Problem 7: Show that the formula is invariant under translation and rotation, and that it therefore suffices to show its validity when $A = 0$ and B is real.

Problem 9: Take the origin at P and show that MN is LN rotated around N through $60°$.

Problem 10: The centroid of $\triangle ABC$ is $\frac{1}{3}(A + B + C)$.

Problem 14: Take $A = 0$, $B = 1$ and $C = 2w$, where $w = e^{60° i}$. Then $D + \overline{D} = 2$, $E/w = \overline{E}/\overline{w}$ and $F = \overline{F}$. Find the side length $\triangle DEF$ in terms of F.

Problem 15: Calculate the complex number P, and also P', which satisfies $[ABP'] = [CDP']$. Try to factorize $P - P'$.

Problem 16: Choose $k_1 = \kappa(1, r)$, $k_2 = \kappa(0, 1)$, $k_3 = \kappa(Q, s)$. Calculate Q in terms of r and s. You need to prove that $\operatorname{Re} G = \operatorname{Re} H = 1 - r$.

INDEX