2019 Wits Mathematics Competition
Final Round
Undergraduate
Time: 90 Minutes

Instructions

This exam consists of 12 questions. The first 10 are single answer and are worth 3 marks each. The last 2 are proof questions which require full solutions. They are out of 10 marks each.

"A mathematician is a device for turning coffee into theorems." — Paul Erdos

**Full Name:**
School
Division:	**Undergraduate**
Grade
E-mail

Answer Section A below

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A. Single Answer Question

1. How many prime numbers are there between 1 and 100?

2. How many positive perfect squares less than 10^7 (ten million) are multiples of 54?

3. How many four digit numbers contain either a 5 or a 6 (or both)?

4. Compute $\int_{-5}^{5} \sqrt{25 - x^2} dx$.

5. A function f is defined such that $f(1) = 2020$ and for $n > 1$, $f(1) + f(2) + .. + f(n) = n^2 f(n)$. Compute $f(2019)$

6. Let $A = a, b, c, d, e, f$ and let P be the set of all nonempty subsets of A. A function f from P to A is a "selector" function if $f(B)$ is in B, and $f(B \bigcup C)$ is either equal to $f(B)$ or $f(C)$. How many selector functions are there?

7. Seven marked points $P_1, P_2, P_3, P_4, P_5, P_6$ and P_7 lie on a line in that order. A marked point P is placed on the line and the quantity S is calculated as the sum of the lengths of the seven line segments $\overline{PP_1}$, $\overline{PP_2}$,$\overline{PP_3}$,$\overline{PP_4}$,$\overline{PP_5}$,$\overline{PP_6}$ and$\overline{PP_7}$. Describe the set of choices for P such that S is minimal.

8. Identical regular pentagons are placed together side by side in the manner shown. The diagram shows 2 pentagons. How many are required to make a full ring?
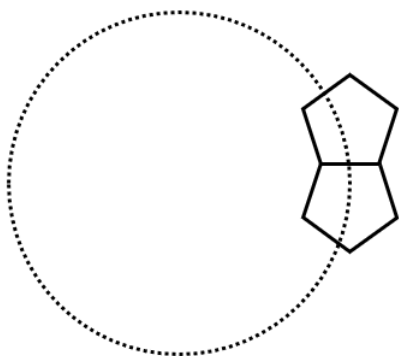
9. Grandma Mathematics is 81 years old and has 4 grandchildren. If the sum of the grandchildrens age's is multiplied by the grandmother's age it gives a 4-digit number whose digits are the ages of her grandchildren. Find the 4 digit number.

10. Find a 6-digit number N such that the numbers $2N$, $3N$, $4N$, $5N$ and $6N$ consist of the same digits as N but written in a different order.

B. Proof Questions

11. 2019 people are about to board a plane with 2019 seats numbered 1 through 2019. They have tickets numbered by the integers 1 through 2019 inclusive. The first person enters the plane and instead of sitting in seat 1 chooses a seat at random. The rest of the passengers are better behaved, and they will sit in there assigned seat if possible. Otherwise they will sit in a random empty seat. What is the probability that passenger 2019 sits in seat 2019?

12. (a) Sixteen players enter a chess contest. They each have a different skill level and for purposes of this question a better player will always beat a weaker player. The contest takes the format of a 4 round knockout tournament, where the players are set against each other randomly. Find the probability that the third best player is eliminated by the best.

(b). How does this differ if 256 players play in an 8 round knockout tournament?

(c). If 2^n players compete in an n round tournament?